SOLUTION OF APERTURE DISTRIBUTION FOR A GIVEN RADIATION FIELD PATTERN

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# SOLUTION OF APERTURE DISTRIBUTION FOR A GIVEN RADIATION FIELD PATTERN

By

Donald Vance Cox

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Dedication

This essay is dedicated to my wife.

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# Chapter I

## Introduction

1.1 As a result of growing interest in microwave electromagnetic radiation, considerable research has been directed toward the shaping of the secondary radiation patterns. Since the secondary, or distant field, pattern depends upon the radiating source, the problem is to correctly determine this source. Experimental selection of the source required for a desired secondary pattern is expensive in both time and money, and these difficulties have led to the quest for an analytic solution.

1.2 The mathematical relationship of the secondary pattern to its source has long been known*, but the relationship is an integral equation** in which the source function appears in the integrand. Several attempts to solve this integral equation have resulted in partial success. The late Commander Dave Johnston, Jr., U.S. Navy, prepared an excellent survey of the problem and an approach to its solution (3) which formed the initial stimulus for this paper. A.C. Spence (9,10), P.M. Woodward and J.D. Lawson (11), and S.Silver(8) have also investigated the analytic problem; but they have used approaches which extend the limits of the aperture to infinity, something which cannot always be done.

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* Silver, Microwave Antenna Theory and Design, (8), pp 104.

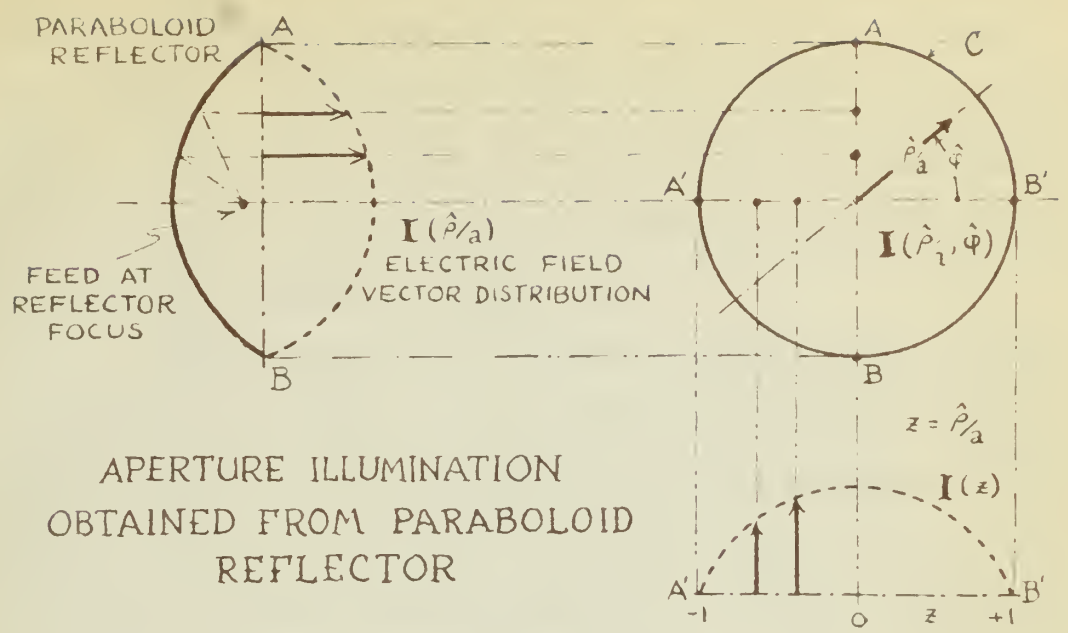
** $$\bar{E}(x) = \bar{G} a^2 \int_0^1 z\, I(z)\, J_0(xz)\, dz .$$

(In this paper, functions will be indicated by roman letters. Numbers in parentheses indicate references listed in the Bibliography. Vectors will be indicated by a horizontal line over a letter.)

$$E(x) = \int_0^{\infty} I(t)\, J_0(xt)\, dt .$$

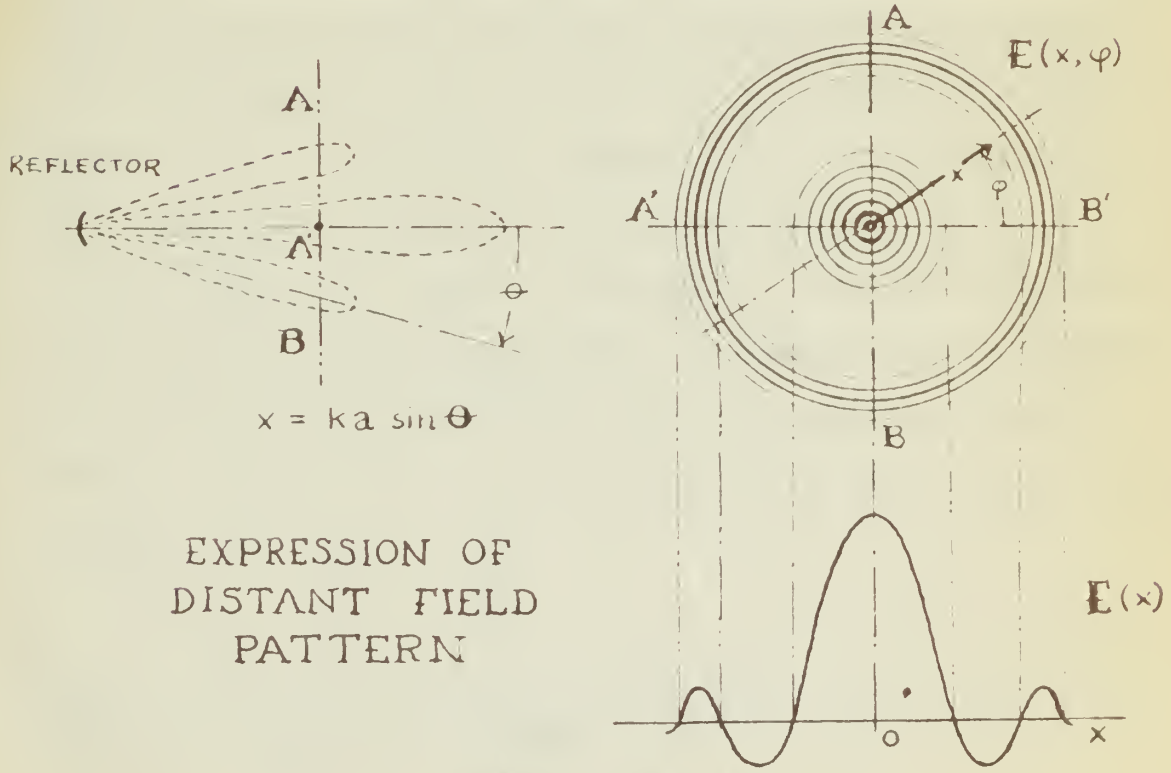
1.3 For the purposes of this paper, the radiation source is defined to consist of a circularly symmetric distribution of electric and magnetic field vectors over a finite circular aperture. This source can be readily correlated to the physical electromagnetic radiation source obtained from a paraboloid reflector with the antenna feed at the focus of the reflector (Fig. 1a). The source can then be considered as the distribution of electric and magnetic vectors in the plane containing the rim of the paraboid reflector (AB in Fig. 1a) with the aperture defined by the reflector rim. Since just circularly symmetric sources are to be considered, the amplitude distributions of electric and magnetic vectors depend only on the distance from the axis of the reflector. The distant field patterns are maps of the electric and magnetic field vector distribution in distant space. If amplitudes of these vectors were to be measured in a plane in distant space normal to the extended axis of the reflector, a distant field pattern would be obtained (Fig. 1b). And since the distant field pattern is circularly symmetric for a circularly symmetric source distribution, this distant field pattern may be completely expressed as a function of one variable, the distance from the reflector axis (Fig. 1b).

1.4 An analytic solution of the amplitude distribution of electric and magnetic field vectors over a finite aperture necessary to produce a given secondary radiation pattern is presented in this paper. This solution, rigorous and requiring no assumptions regarding aperture limits, is then developed into two methods for practical determination of the aperture illumination required for a given distant field pattern. The only restriction placed on either method is that the distant field pattern considered must be of such nature that it may be set up by circularly symmetric, plane polarized waves over the aperture.




APERTURE ILLUMINATION OBTAINED FROM PARABOLOID REFLECTOR

FIGURE 1a




EXPRESSION OF DISTANT FIELD PATTERN

FIGURE 1b

1.5 Chapter II contains a derivation of the integral equation relating the aperture distribution and the distant field pattern. This equation of relationship is the foundation of the later methods.

1.6 Chapter III presents a method for solution of the aperture distribution for use when the distant field pattern can be explicitly expressed as a summation of V functions.*

1.7 Chapter IV presents a method for solution which utilizes the developments of Chapter III but which does not require that the distant field pattern be explicitly expressed in V functions. The method is based on the use of a mathematical device and suggests a new type of transform relationship between the aperture illumination pattern and the distant field pattern.

1.8 Conclusions and summary of methods are presented in Chapter V.

1.9 Appendix I contains a list of symbols used in the paper. These are also explained in footnotes throughout the paper where the author considered such explanation expedient. Appendix II presents definitions of the mathematical functions used in the developments and a collection of data on these functions obtained in the course of the investigation. Examples of the use of the method proposed in Chapter III are given in Appendix III, and examples of the method of Chapter IV appear in Appendix IV. Appendix V contains tables and curves. Appendix VI contains two important later developments in the theory.

---

* $V_n(x) = \frac{\Lambda_n(x)}{n}$ , (Appendix II).

## Chapter II

### Derivation of Distant Field Equation

2.1 For a complete understanding of the relationship between the aperture distribution and the distant field radiation pattern, it is first necessary to establish the correlating equation. The development here presented is that of J.A.Stratton and Dr. L.J.Chu* and of S.A.Schelkunoff**. Much of the derivation and application was discussed by the late Commander Dave Johnston, U.S.N., in his essay (3) submitted to the Johns Hopkins University in 1949. This method consists of direct integration of Maxwell's field equations by the use of the vector analogue of Green's theorem and the transformation of the result to analytic form.

2.2 The vector analogue of Green's theorem is

$$\int_V (\bar{Q}\cdot\bar{\nabla}\times\bar{\nabla}\times\bar{P} - \bar{P}\cdot\bar{\nabla}\times\bar{\nabla}\times\bar{Q})\,dv = \int_S (\bar{P}\times\bar{\nabla}\times\bar{Q} - \bar{Q}\times\bar{\nabla}\times\bar{P})\cdot\bar{n}\,da, \qquad 2.01^{***}$$

where $\bar{P}$ and $\bar{Q}$ are vector functions of position continuous and with continuous first and second derivatives throughout V and on S. V is

---

* Electromagnetic Theory (1), sections 4-14, 8-14, 8-15.

** Electromagnetic Waves (2), Chapter IX.

*** Green's theorem in the form of (2.01) is obtained by applying the divergence theorem to the vector $\bar{P}\times(\bar{\nabla}\times\bar{Q})$:

$$(a)\quad \int_V \left[\bar{\nabla}\cdot\{\bar{P}\times(\bar{\nabla}\times\bar{Q})\}\right] dv = \int_S \left[\{\bar{P}\times(\bar{\nabla}\times\bar{Q})\}\cdot\bar{n}\right] da.$$

Expanding the volume integrand of (a), we have

$$(b)\quad \int_V \left[\{(\bar{\nabla}\times\bar{Q})\cdot\bar{\nabla}\times\bar{P}\} - \{\bar{P}\cdot\bar{\nabla}\times(\bar{\nabla}\times\bar{Q})\}\right] dv = \int_S \left[\{\bar{P}\times(\bar{\nabla}\times\bar{Q})\}\cdot\bar{n}\right] da.$$

Interchanging the roles of $\bar{P}$ and $\bar{Q}$ and subtracting the resultant equation from (b), we obtain (2.01).

$$\int_{\Omega} [\underline{a}\cdot\{\underline{b}\times(\underline{c}\times\underline{d})\}]\,d\Omega = \int_{\Omega} [\{\underline{b}\times(\underline{a}\times\underline{c})\}\cdot\underline{n}]\,d\Gamma .$$

$$\int_{\Omega} [\{(\underline{a}\times\underline{c})\cdot\underline{d}\times\underline{b}\} - \{\underline{b}\cdot\underline{a}\times(\underline{b}\times\underline{c})\}]\,d\Omega = \int_{\Gamma} [\{\underline{b}\times(\underline{a}\times\underline{c})\}\cdot\underline{n}]\,d\Gamma .$$

$$\underline{b}\times(\underline{a}\times\underline{c}) :$$

$$\int_{\Omega} [\{(\underline{a}\times\underline{c})\cdot\underline{d}\times\underline{b}\} - \{\underline{b}\cdot\underline{a}\times(\underline{b}\times\underline{c})\}]\,d\Omega = \int_{\Gamma} [\{\underline{b}\times(\underline{a}\times\underline{c})\}\cdot\underline{n}]\,d\Gamma ,$$

$$\int_{\Omega} [\{(\underline{a}\times\underline{c})\cdot\underline{d}\times\underline{b}\} - \{\underline{b}\cdot\underline{a}\times(\underline{b}\times\underline{c})\}]\,d\Omega = \int_{\Gamma} [\underline{a}\cdot\{\underline{b}\times(\underline{a}\times\underline{c})\}\cdot\underline{n}]\,d\Gamma .$$

a closed region of space bounded by a regular surface S.

2.5 Maxwell's field equations may be written in the form:*

$$\bar{\nabla}\times\bar{E} - i\omega\mu\bar{H} = -\bar{J}^*, \qquad 2.02$$

$$\bar{\nabla}\times\bar{H} + i\omega\varepsilon\bar{E} = \bar{J}, \qquad 2.03$$

$$\bar{\nabla}\cdot\bar{H} = \frac{1}{\mu}\rho^*, \qquad 2.04$$

$$\bar{\nabla}\cdot\bar{E} = \frac{1}{\varepsilon}\rho. \qquad 2.05$$

Let us assume that time is contained in the electric and magnetic field vectors, $\bar{E}$ and $\bar{H}$, only as a factor $e^{i\omega t}$ and that at every point in space the field vectors are subject to these equations. The equations of continuity, obtained by taking the divergence of (2.02) and (2.03), relate the real and fictitious** currents and charges.

$$\bar{\nabla}\cdot\bar{J} - i\omega\rho = 0, \qquad 2.06$$

$$\bar{\nabla}\cdot\bar{J}^* - i\omega\rho^* = 0. \qquad 2.07$$

---

* Symbols used in the development (also listed in Appendix I):

| | |
|---|---|
| $\bar{E}$ | vector intensity of the electric field |
| $\bar{H}$ | vector intensity of the magnetic field |
| $\bar{J}$ | electric current density |
| $\rho$ | electric charge density |
| $\bar{J}^*$ | fictitious magnetic current density** |
| $\rho^*$ | fictitious magnetic charge density** |
| $\mu$ | permeability of space |
| $\varepsilon$ | permittivity of space |
| $\omega$ | angular velocity |
| $k$ | phase constant |

** $\bar{J}^*$ and $\rho^*$ are fictitious quantities having no known physical existence. In the development here presented, arbitrary discontinuities in $\bar{E}$ and $\bar{H}$ will be assumed. Actually a physical impossibility, these discontinuities would be generated by physically existent surface distributions of magnetic current and charge.

$$\nabla \cdot \vec{E} = \frac{1}{\varepsilon_0} \rho ,$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \rho^* ,$$

$$\nabla \times \vec{H} + i\omega\varepsilon \vec{E} = \vec{J} ,$$

$$\nabla \times \vec{E} - i\omega\mu_0 \vec{H} = -\vec{J}^* .$$

$$\nabla \cdot \vec{J} - i\omega\rho = 0 ,$$

$$\nabla \cdot \vec{J}^* - i\omega\rho^* = 0 .$$

2.4 The vector $\bar{E}$ satisfies

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) - k^2\bar{E} = i\omega\mu\bar{J} - \bar{\nabla}\times\bar{J}^*. \qquad 2.08$$

Equation (2.08) is obtained by taking the curl of (2.02),

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) - i\omega\mu(\bar{\nabla}\times\bar{H}) = -\bar{\nabla}\times\bar{J}^*, \qquad 2.09$$

and substituting from (2.03),

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) - i\omega\mu(\bar{J} - i\omega\varepsilon\bar{E}) = -\bar{\nabla}\times\bar{J}^*. \qquad 2.10$$

Now (2.10) may be written

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{E}) - i\omega\mu\bar{J} - \omega^2\varepsilon\mu\bar{E} = -\bar{\nabla}\times\bar{J}^*, \qquad 2.11$$

which is the same as (2.08) since $k^2 = \omega^2\varepsilon\mu$.

2.5 The vector $\bar{H}$ satisfies

$$\bar{\nabla}\times(\bar{\nabla}\times\bar{H}) - k^2\bar{H} = i\omega\varepsilon\bar{J}^* + \bar{\nabla}\times\bar{J}, \qquad 2.12$$

which is obtained by taking the curl of (2.03) and substituting from (2.02) in the manner of paragraph (2.4).

2.6 In (2.01) we let $\bar{P} \equiv \bar{E}$ and $\bar{Q} \equiv \Phi\bar{a}$, where $\bar{a}$ is a unit vector in an arbitrary direction and $\Phi = \frac{e^{-ikr}}{r}$ *. The distance r, measured from the element at (X,Y,Z) to the point of observation at (X', Y', Z') may be written (Figure 3)

$$r = \sqrt{(X'-X)^2 \ (Y'-Y)^2 \ (Z'-Z)^2}. \qquad 2.13$$

Since $\Phi = \frac{e^{-ikr}}{r}$, 2.14

$$\bar{\nabla}\Phi = -\Phi\left(ik + \frac{1}{r}\right)\bar{r}_o, \qquad 2.15$$

and $$\bar{\nabla}\cdot\bar{\nabla}\Phi = -k^2\Phi, \qquad 2.16$$

---

* Stratton (1) uses $e^{ikr}$ for the positive traveling wave. Schelkunoff (2), as do most engineers, uses $e^{-ikr}$. The latter is used in this work.

$$\nabla \cdot \nabla \times (\vec{E} - \kappa' \vec{E} = i\omega\mu_0 \vec{J} - \vec{J} \times \vec{J}^*.$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - i\omega\mu_0 (\vec{\nabla} \times \vec{H}) = -\vec{\nabla} \times \vec{J}^*,$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - i\omega\mu_0 (\vec{J} - i\omega\varepsilon \vec{E}) = -\vec{\nabla} \times \vec{J}^*.$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) - i\omega\mu_0 \vec{J} - \omega^2 \varepsilon_0 \mu \vec{E} = -\vec{\nabla} \times \vec{J}^*,$$

$$\kappa' = \omega^2 \varepsilon_0 \mu.$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) - \kappa'^2 \vec{H} = i\omega\varepsilon \vec{J}^* + \vec{\nabla} \times \vec{J}^*,$$

$$\vec{\Phi} = \frac{e^{-ikr}}{r}$$

$$= \sqrt{\qquad\qquad - \qquad\qquad}.$$

$$\vec{\Phi} = \frac{1}{r} e^{-ikr},$$

$$\vec{\nabla} \cdot \vec{\nabla} \vec{\Phi} = -\kappa'^2 \vec{\Phi},$$

$$\vec{\nabla} \vec{\Phi} = -\vec{\Phi} (ik + \frac{1}{r}) \vec{r}.$$

$$e^{-ikr}$$

we have the identities:

a. $\bar{\nabla}\times\bar{Q} = \bar{\nabla}\times\Phi\bar{a} = \bar{\nabla}\Phi\times\bar{a}$ , (2.17)

b. $\bar{\nabla}\times\bar{\nabla}\times\bar{Q} = \bar{\nabla}\times\bar{\nabla}\Phi\times\bar{a}$

$= \bar{\nabla}\Phi(\bar{\nabla}\cdot\bar{a}) - \bar{a}(\bar{\nabla}\cdot\bar{\nabla}\Phi) + (\bar{a}\cdot\bar{\nabla})\bar{\nabla}\Phi - (\bar{\nabla}\Phi\cdot\bar{\nabla})\bar{a}$

$= \bar{a}k^2\Phi + (\bar{a}\cdot\bar{\nabla})\bar{\nabla}\Phi$ , (2.18)

(Since $\bar{a}$ is a constant vector, $\bar{\nabla}\cdot\bar{a} = 0$ and $(\bar{\nabla}\Phi\cdot\bar{\nabla})\bar{a} = 0$ .)

c. $\bar{\nabla}\times\bar{P} = \bar{\nabla}\times\bar{E}$ , (2.19)

d. $\bar{\nabla}\times\bar{\nabla}\times\bar{P} = \bar{\nabla}\times\bar{\nabla}\times\bar{E}$

$= k^2\bar{E} + i\omega\mu\bar{J} - \bar{\nabla}\times\bar{J}^*$, from (2.6). (2.20)

2.7 Expressing (2.01) in terms of the identities of paragraph (2.6), we obtain

$$\int_V \left\{\Phi\bar{a}\cdot\left[k^2\bar{E} + i\omega\mu\bar{J} - \bar{\nabla}\times\bar{J}^*\right] - \bar{E}\cdot\left[\bar{a}k^2\Phi + \bar{\nabla}(\bar{a}\cdot\bar{\nabla}\Phi)\right]\right\} dv$$

$$= \int_S \left\{\left[\bar{E}\times\bar{\nabla}\Phi\times\bar{a}\right] - \left[\Phi\bar{a}\times\bar{\nabla}\times\bar{E}\right]\right\}\cdot\bar{n}\, da. \qquad (2.21)$$

Now let us consider the volume integral of (2.21),

$$\int_V \left\{\Phi\bar{a}\cdot\left[k^2\bar{E} + i\omega\mu\bar{J} - \bar{\nabla}\times\bar{J}^*\right] - \bar{E}\cdot\left[\bar{a}k^2\Phi + \bar{\nabla}(\bar{a}\cdot\bar{\nabla}\Phi)\right]\right\} dv,$$

which becomes

$$\bar{a}\cdot\int_V \left[i\omega\mu\bar{J}\Phi - \bar{\nabla}\times\bar{J}^*\Phi\right] dv - \int_V \left[\bar{E}\cdot\bar{\nabla}(\bar{a}\cdot\bar{\nabla}\Phi)\right] dv .$$

Applying the vector identity, $\bar{A}\cdot\bar{\nabla}B = \bar{\nabla}\cdot B\bar{A} - B\bar{\nabla}\cdot\bar{A}$ ,

(with $\bar{A} = \bar{a}$ and $B = \{\bar{a}\cdot\bar{\nabla}\Phi\}$ ) to the last integral above, the volume integral of (2.21) may be written

$$\bar{a}\cdot\int_V \left[i\omega\mu\bar{J}\Phi - \bar{\nabla}\times\bar{J}^*\Phi\right] dv - \int_V \bar{\nabla}\cdot(\bar{a}\cdot\bar{\nabla}\Phi)\bar{E}\, dv + \int_V (\bar{a}\cdot\bar{\nabla}\Phi)(\bar{\nabla}\cdot\bar{E})\, dv.$$

$$\int_{\hat{q}\cdot}\left[i\omega\mu_1\vec{\phi}-\vec{\Delta}\times\vec{J}_*\phi\right]dq-\int_{\hat{q}}\vec{\Delta}\cdot(\vec{q}\cdot\vec{\Delta}\phi)\vec{E}\,dq+\int_{\hat{q}}(\vec{q}\cdot\vec{\Delta}\phi)(\vec{\Delta}\cdot\vec{E})dq.$$

$$\vec{L}=\vec{L}_{\cdot}=\{\vec{L}_{\cdot}\cdot\vec{\Delta}\phi\}$$

$$\vec{V}\cdot\vec{\Delta}R=\vec{\Delta}\cdot R\vec{V}-R\vec{\Delta}\cdot\vec{V}\ ,$$

$$\int_{\hat{q}\cdot}\left[i\omega\mu_1\vec{J}\phi-\vec{\Delta}\times\vec{J}_*\phi\right]dq-\int_{\hat{q}}\left[\vec{E}\cdot\vec{\Delta}(\vec{\pi}\cdot\vec{\Delta}\phi)\right]dq\ .$$

$$\int_{\hat{q}}\left\{\phi\vec{q}\cdot\left[k_1^2\vec{E}+i\omega\mu_1\vec{J}-\vec{\Delta}\times\vec{J}_*\right]-\vec{E}\cdot\left[\vec{q}k_1^2\phi+\vec{\Delta}(\vec{q}\cdot\vec{\Delta}\phi)\right]\right\}dq'$$

$$=\int_{?}\left\{\left[\vec{E}\times\vec{\Delta}\phi\times\vec{q}\right]-\left[\phi\vec{q}\times\vec{\Delta}\times\vec{E}\right]\right\}\cdot\mu\ dq'.$$

$$\int_{\hat{q}}\left\{\phi\vec{q}\cdot\left[k_1^2\vec{E}+i\omega\mu_1\vec{J}-\vec{\Delta}\times\vec{J}_*\right]-\vec{E}\cdot\left[\vec{q}k_1^2\phi+\vec{\Delta}(\vec{q}\cdot\vec{\Delta}\phi)\right]\right\}dq$$

$$=k_1^2\vec{E}+i\omega\mu\vec{J}-\vec{\Delta}\times\vec{J}_*\ ,$$

$$\vec{\Delta}\times\vec{\Delta}\times\vec{b}=\vec{\Delta}\times\vec{\Delta}\times\vec{E}$$

$$\vec{\Delta}\times\vec{b}=\vec{\Delta}\times\vec{E}\ ,$$

$$(\vec{\Delta}\phi\cdot\vec{\Delta})\vec{q}=0\qquad\vec{\Delta}\cdot\vec{q}=0$$

$$=3k_1^2\phi+(\vec{q}\cdot\vec{\Delta})\vec{\Delta}\phi\ ,$$

$$=\vec{\Delta}\phi(\vec{\Delta}\cdot\vec{q})-\vec{q}(\vec{\Delta}\cdot\vec{\Delta}\phi)+(\vec{q}\cdot\vec{\Delta})\vec{\Delta}\phi-(\vec{\Delta}\phi\cdot\vec{\Delta})\vec{q}$$

$$\vec{\Delta}\times\vec{\Delta}\times\vec{\phi}=\vec{\Delta}\times\vec{\Delta}\phi\times\vec{q}$$

$$\vec{\Delta}\times\vec{\zeta}=\vec{\Delta}\times\phi\vec{q}=\vec{\Delta}\phi\times\vec{q}\ .$$

By applying the divergence theorem to the vector $(\bar{a}\cdot\bar{\nabla}\Phi)\bar{E}$ in the second integral above and substituting from (2.05) in the third integral, we obtain

$$\bar{a}\cdot\int_V\left[i\omega\mu\bar{J}\Phi-\bar{\nabla}\times\bar{J}^*\Phi\right]dv-\int_S\left[(\bar{a}\cdot\bar{\nabla}\Phi)\bar{E}\cdot\bar{n}\right]da+\int_V\frac{\rho}{\varepsilon}(\bar{a}\cdot\bar{\nabla}\Phi)\,dv.$$

Now let us consider the vector integration formula

$$\int_V\left[\bar{\nabla}\times\Phi\bar{J}^*\right]dv=\int_S\left[\bar{n}\times\bar{J}^*\Phi\right]da,$$

or $$\int_V\left[(\bar{\nabla}\times\bar{J}^*)\Phi-(\bar{J}^*\times\bar{\nabla}\Phi)\right]dv=\int_S\left[\bar{n}\times\bar{J}^*\Phi\right]da,$$

which may be written

$$\int_V\left[\bar{\nabla}\times\bar{J}^*\Phi\right]dv=\int_V\left[\bar{J}^*\times\bar{\nabla}\Phi\right]dv+\int_S\left[\bar{n}\times\bar{J}^*\Phi\right]da.$$

Application of this formula allows expression of the volume integral of (2.21) as

$$\int_V\left\{\Phi\bar{a}\cdot\left[k^2\bar{E}+i\omega\mu\bar{J}-\bar{\nabla}\times\bar{J}^*\right]-\bar{E}\cdot\left[\bar{a}k^2\Phi+\bar{\nabla}(\bar{a}\cdot\bar{\nabla}\Phi)\right]\right\}dv$$

$$=\bar{a}\cdot\int_V\left[i\omega\mu\bar{J}\Phi-\bar{J}^*\times\bar{\nabla}\Phi+\frac{\rho}{\varepsilon}\bar{\nabla}\Phi\right]dv-\bar{a}\cdot\int_S\left[(\bar{E}\cdot\bar{n})\bar{\nabla}\Phi+(\bar{n}\times\bar{J}^*\Phi)\right]da. \qquad 2.22$$

2.9 Now let us consider the surface integral of (2.21):

$$\int_S\left\{\left[\bar{E}\times\bar{\nabla}\Phi\times\bar{a}\right]-\left[\Phi\bar{a}\times\bar{\nabla}\times\bar{E}\right]\right\}\cdot\bar{n}\,da$$

$$=\int_S\left[\bar{\nabla}\Phi(\bar{E}\cdot\bar{a})-\bar{a}(\bar{E}\cdot\bar{\nabla}\Phi)-\bar{\nabla}(\Phi\bar{a}\cdot\bar{E})+\bar{E}(\Phi\bar{a}\cdot\bar{\nabla})\right]\cdot\bar{n}\,da,$$

$$=\bar{a}\cdot\int_S\left[\bar{E}(\bar{\nabla}\Phi\cdot\bar{n})-\bar{n}(\bar{\nabla}\Phi\cdot\bar{E})-\Phi\left\{\bar{E}(\bar{n}\cdot\bar{\nabla})-\bar{\nabla}(\bar{n}\cdot\bar{E})\right\}\right]da,$$

$$=\bar{a}\cdot\int_S\left[\left\{\bar{\nabla}\Phi\times(\bar{E}\times\bar{n})\right\}-\Phi\left\{\bar{n}\times(\bar{E}\times\bar{\nabla})\right\}\right]da,$$

$$=\bar{a}\cdot\int_S\left[\left\{(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi\right\}+\Phi\left\{\bar{n}\times(\bar{\nabla}\times\bar{E})\right\}\right]da,$$

and substituting from (2.02),

$$= \mu_0 \left\{ \left[ (\mu \times E) \times \hat{A}\Phi \right] + \Phi \left[ \mu \times (\hat{A} \times E) \right] \right\} dv' ,$$

$$= \mu_0 \left\{ \left[ \hat{A}\Phi \times (E \times \mu) \right] - \Phi \left[ \mu \times (E \times \hat{A}) \right] \right\} dv' ,$$

$$= \mu_0 \left\{ E \left( \hat{A}\Phi \cdot \mu \right) - \mu (\hat{A}\Phi \cdot E) - \Phi \left[ E(\mu \cdot \hat{A}) - \hat{A}(\mu \cdot E) \right] \right\} dv' ,$$

$$= \int \left[ \hat{A}\Phi (E \cdot \mu) - \mu (E \cdot \hat{A}\Phi) - \hat{A}(\nabla' \cdot E) + E(\Phi \nabla \cdot \hat{A}) \right] \cdot \mu \, dv' ,$$

$$\int \left[ E \times \hat{A}\Phi \times \nabla \right] - \left[ \Phi \nabla \times \hat{A} \times E \right] \cdot \mu \, dv$$

$$= \mu_0 \int \left[ i\omega \mu_1 \Phi - \nabla_1 \times \hat{A}\Phi + \frac{1}{\varepsilon} \hat{A}\Phi \right] dA - \mu_0 \int \left[ (E \cdot \mu) \hat{A}\Phi + (\mu \times \nabla_1 \Phi) \right] dv' .$$

$$\int \left\{ \Phi \nabla \cdot \left[ \varepsilon_r E + i\omega \mu_1 \nabla_1 \times \nabla_1 \times \right] - E \cdot \left[ \nabla r_s \Phi + \hat{A} (\nabla \cdot \hat{A}\Phi) \right] \right\} dv$$

$$\int \left[ \hat{A} \times \nabla_1 \Phi \right] dA = \int \left[ \nabla_1 \times \hat{A}\Phi \right] dA + \int \left[ \mu \times \nabla_1 \times \hat{E} \right] dv' .$$

$$\int \left[ (\hat{A} \times \nabla_1) \Phi - (\nabla_1 \times \hat{A}\Phi) \right] dA = \int \left[ \mu \times \nabla_1 \Phi \right] dv' ,$$

$$\int \left[ \hat{A} \times \Phi \nabla_1 \right] dA = \int \left[ \mu \times \nabla_1 \Phi \right] dv' ,$$

$$\mu_0 \int \left[ i\omega \mu_1 \hat{E} - \hat{A} \times \nabla_1 \Phi \right] dA - \int \left[ (\nabla \cdot \hat{A}\Phi) E \cdot \mu \right] dv' + \int \frac{1}{\varepsilon} (\nabla \cdot \hat{A}\Phi) \, dA' .$$

$$\nabla \cdot \hat{A}\Phi$$

$$= \bar{a}\cdot\int_S [\{(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi\} + \Phi\{\bar{n}\times(i\omega\mu\bar{H} - \bar{J}^*)\}]\,da,$$

$$= \bar{a}\cdot\int_S [\{(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi\} + i\omega\mu(\bar{n}\times\bar{H})\Phi - (\bar{n}\times\bar{J}^*\Phi)]\,da. \qquad 2.23$$

Combining (2.22) and (2.23), remembering that $\bar{a}$ is an arbitrary unit vector, we obtain:

$$\int_V [i\omega\mu\bar{J}\Phi - (\bar{J}^*\times\bar{\nabla}\Phi) + \frac{\rho}{\varepsilon}\bar{\nabla}\Phi]\,dv$$

$$= \int_S [\{(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi\} + i\omega\mu(\bar{n}\times\bar{H})\Phi + (\bar{E}\cdot\bar{n})\bar{\nabla}\Phi]\,da. \qquad 2.24$$

2.8 It must now be noted that $\bar{\nabla}\Phi = \Phi\bar{r}$ has a singularity at $r = 0$. To exclude this point, circumscribe a sphere of radius $r_1$, whose surface is $S_i$, about this point $(X', Y', Z')$. The normal $\bar{n}$ is directed out of V and hence radially toward $(X', Y', Z')$. $\bar{r}_o$ is a unit vector directed from the source toward $(X', Y', Z')$. Hence on the sphere, $\bar{n} = \bar{r}_o$. Now the volume V is bounded internally by the surface $S_i$ and externally by S. Therefore, the surface integral of (2.24) must be considered over $S + S_i$:

$$\int_V [i\omega\mu\bar{J}\Phi - (\bar{J}^*\times\bar{\nabla}\Phi) + \frac{\rho}{\varepsilon}\bar{\nabla}\Phi]\,dv$$

$$= \int_{S+S_i} [\{(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi\} + i\omega\mu(\bar{n}\times\bar{H})\Phi + (\bar{E}\cdot\bar{n})\bar{\nabla}\Phi]\,da. \qquad 2.25$$

Let us consider the surface integral over $S_i$. The integral over $S_i$ of (2.25) may be written

$$\frac{e^{-ikr_1}}{r_1^2}\int_{S_i}\bar{E}\,da + ik\frac{e^{-ikr_1}}{r_1}\int_{S_i}\bar{E}\,da + i\omega\mu\frac{e^{-ikr_1}}{r_1}\int_{S_i}(\bar{n}\times\bar{H})\,da, \qquad 2.26$$

because of the vector identity

$$(\bar{n}\times\bar{E})\times\bar{n} + (\bar{n}\cdot\bar{E})\,\bar{n} = \bar{E}$$

$$(\underline{\mu}\times\underline{E})\times\underline{\mu} + (\underline{\mu}\cdot\underline{E})\,\underline{\mu} = \underline{E}$$

$$\frac{L_2'}{e^{-ikL_2'}}\int_{S_2'} E\,dq + ik\frac{L_2'}{e^{-ikL_2'}}\int_{S_2'} E\,dq + i\omega\mu_r\frac{L_2'}{e^{-ikL_2'}}\int_{S_2'}(\underline{\mu}\times\underline{H})\,dq,$$

$$= \int_{S_2'}\left[(\underline{\mu}\times\underline{E})\times\Delta\Phi\right] + i\omega\mu_r(\underline{\mu}\times\underline{H})\Phi + (\underline{E}\cdot\underline{\mu})\Delta\Phi\,]\,dq.$$

$$\int\left[i\omega\mu_r\underline{J}\Phi - (\underline{J}_*\times\Delta\Phi) + \frac{i}{k}\Delta\Phi\right]dq$$

$$= \int\left[(\underline{\mu}\times\underline{E})\times\Delta\Phi\right] + i\omega\mu_r(\underline{\mu}\times\underline{H})\Phi + (\underline{E}\cdot\underline{\mu})\Delta\Phi\,]\,dq.$$

$$\int\left[i\omega\mu_r\underline{J}\Phi - (\underline{J}_*\times\Delta\Phi) + \frac{i}{k}\Delta\Phi\right]dq$$

$$= \int\left[(\underline{\mu}\times\underline{E})\times\Delta\Phi\right] + i\omega\mu_r(\underline{\mu}\times\underline{H})\Phi - (\underline{\mu}\times\underline{J}_*\Phi)\right]dq.$$

$$= \int\left[(\underline{\mu}\times\underline{E})\times\Delta\Phi + ik(\underline{\mu}\times\underline{H})\Phi - (\underline{\mu}\times\underline{J}_*\Phi)\right]dq,$$

and because, on $S_i$*,

$$\bar{\nabla}'\Phi = \frac{e^{-ikr}}{r}\left(\frac{1}{r}+ik\right)\bar{r}_o = \frac{e^{-ikr}}{r}\left(\frac{1}{r}+ik\right)\bar{n}\,.$$

Let $\underline{\bar{E}}$ and $\underline{(\bar{n}\times\bar{H})}$ denote the mean values of vectors $\bar{E}$ and $(\bar{n}\times\bar{H})$ over $S_i$. The surface integrals over $S_i$ then have the value

$$4\pi r_i^2\left(\underline{\bar{E}}\left\{\frac{e^{-ikr_i}}{r_i^2}+ik\frac{e^{-ikr_i}}{r_i}\right\}+i\omega\mu\,\frac{e^{-ikr_i}}{r_i}\{\underline{\bar{n}\times\bar{H}}\}\right).$$

In the limit as $r_i \to 0$, this value reduces to $4\pi\bar{E}(X',Y',Z')$. Hence (2.25) may be written

$$\bar{E}(X',Y',Z') = \frac{1}{4\pi}\int_V\left[i\omega\mu\bar{J}\Phi-(\bar{J}^*\times\bar{\nabla}\Phi)+\frac{\rho}{\varepsilon}\bar{\nabla}\Phi\right]dv$$

$$-\frac{1}{4\pi}\int_S\left[i\omega\mu(\bar{n}\times\bar{H})\Phi+(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi+(\bar{E}\cdot\bar{n})\bar{\nabla}\Phi\right]da\,, \qquad 2.27$$

expressing the electric field vector at any interior point of V. Interchanging vectors leads to a corresponding expression for $\bar{H}$:

$$\bar{H}(X',Y',Z') = \frac{1}{4\pi}\int_V\left[i\omega\varepsilon\bar{J}^*\Phi+(\bar{J}\times\bar{\nabla}\Phi)+\frac{\rho_\mu^*}{\mu}\bar{\nabla}\Phi\right]dv$$

$$-\frac{1}{4\pi}\int_S\left[i\omega\varepsilon(\bar{n}\times\bar{E})\Phi-(\bar{n}\times\bar{H})\times\bar{\nabla}\Phi-(\bar{n}\cdot\bar{H})\bar{\nabla}\Phi\right]da. \qquad 2.28$$

2.9 If all currents and charges can be enclosed within a sphere of finite radius, the field is regular at infinity and either side of S may be chosen as its "interior".

2.10 Suppose that the charge and current distributions in (2.17) are confined to a thin layer on the surface of S. As the depth of this layer diminishes, the densities may be increased. And in the limit, the volume densities are replaced by surface densities. If

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* $\bar{\nabla}'\Phi = -\bar{\nabla}\Phi$ , where the prime above the gradient operator denotes differentiation with respect to the variables at the point of observation $(X',Y',Z')$. (1), Section 3.6, pp 108.

$$\Delta_2^{\vec{E}} = -\lambda\Gamma \quad \text{(where } \Gamma \text{ is the Green's function for } \Delta_2 \text{ in } \Omega\text{)}$$

$$\vec{H}(x,\lambda,\chi) = \frac{4\pi\lambda}{\Gamma}\int_{\Omega}\left[i\omega\varepsilon_1^*\vec{\Phi} + (\vec{\zeta}\times\vec{\nabla}\vec{\Phi}) + (\vec{\zeta}_{\mathrm{tr}}^*\vec{\Delta}\vec{\Phi}\right]dq\nu$$

$$\vec{E}(x,\lambda,\chi) = \frac{4\pi\lambda}{\Gamma}\int_{\Omega}\left[i\omega\mu_1\vec{\Phi} - (\vec{\zeta}\times\vec{\nabla}\vec{\Phi}) + \frac{\varepsilon}{\varepsilon_1}\vec{\Delta}\vec{\Phi}\right]dq\nu$$

$$-\frac{4\pi}{\Gamma}\int_{\Omega}\left[i\omega\mu(\vec{\mu}\times\vec{H})\vec{\Phi} + (\vec{\mu}\times\vec{E})\times\vec{\nabla}\vec{\Phi} + (\vec{E}\cdot\vec{\mu})\vec{\nabla}\vec{\Phi}\right]dq\nu\ ,$$

$$-\frac{4\pi}{\Gamma}\int_{\Omega}\left[i\omega\varepsilon(\vec{\mu}\times\vec{E})\vec{\Phi} - (\vec{\mu}\times\vec{H})\times\vec{\nabla}\vec{\Phi} - (\vec{\mu}\cdot\vec{H})\vec{\nabla}\vec{\Phi}\right]dq\nu.$$

$$\vec{H}(x,\lambda,\chi) = \frac{4\pi\lambda}{\Gamma}\int_{\Omega}\left[i\omega\varepsilon_1^*\vec{\Phi} + (\vec{\zeta}\times\vec{\nabla}\vec{\Phi}) + (\vec{\zeta}_{\mathrm{tr}}^*\vec{\Delta}\vec{\Phi}\right]dq\nu$$

$$-\frac{4\pi}{\Gamma}\int_{\Omega}\left[i\omega\mu(\vec{\mu}\times\vec{H})\vec{\Phi} + (\vec{\mu}\times\vec{E})\times\vec{\nabla}\vec{\Phi} + (\vec{E}\cdot\vec{\mu})\vec{\nabla}\vec{\Phi}\right]dq\nu\ ,$$

$$\vec{E}(x,\lambda,\chi) = \frac{4\pi\lambda}{\Gamma}\int_{\Omega}\left[i\omega\mu_1\vec{\Phi} - (\vec{\zeta}\times\vec{\nabla}\vec{\Phi}) + \frac{\varepsilon}{\varepsilon_1}\vec{\Delta}\vec{\Phi}\right]dq\nu$$

$$4\pi\lambda\int_{\Omega}\left\{\vec{E}\left(\frac{\Gamma'}{\varepsilon'} + ik\frac{\Gamma'}{\varepsilon'}e^{-ikx}\right) + i\omega\mu\frac{\Gamma'}{\varepsilon'}e^{-ikx}(\vec{\mu}\times\vec{H})\right\}.$$

$$\vec{\Delta}_2\vec{\Phi} = \frac{\Gamma_0}{\varepsilon}\left(\frac{\Gamma_0}{\Gamma} + ik\right)\vec{E}e^{-ikx} = \frac{\Gamma_0}{\varepsilon}e^{-ikx}\left(\frac{\Gamma_0}{\Gamma} + ik\right)\vec{\mu}\ .$$

the region V contains no charge or current within its interior or on S, the field at an interior point becomes

$$\bar{E}(x',y',z') = -\frac{1}{4\pi}\int_S [i\omega\mu(\bar{n}\times\bar{H})\Phi + (\bar{n}\times\bar{E})\times\bar{\nabla}\Phi + (\bar{E}\cdot\bar{n})\bar{\nabla}\Phi]\,da. \qquad 2.29$$

Now let the source distribution of electromagnetic energy exist over an aperture defined as a finite circular area of radius "a" in space. The aperture is shielded in the direction of the negative normal by a perfect absorbent screen of radius "a" through which the electromagnetic energy cannot pass. Consider the geometric center of the aperture as the origin of a spherical coordinate system. The point of observation (X',Y',Z') then becomes $(R,\varphi,\theta)$. (Figure 8) Since only circularly symmetric aperture distributions are now being considered, any distribution on the aperture is a function of only one variable, $\hat{\rho}$, and of the constant aperture radius. The fixed observation point's electric field vector, $\bar{E}(X',Y',Z')$, may be expressed as $\bar{E}(R,\theta)$. Equation (2.29) may now be written:

$$\bar{E}(R,\theta) = -\frac{1}{4\pi}\int_S [i\omega\mu(\bar{n}\times\bar{H})\Phi + (\bar{n}\times\bar{E})\times\bar{\nabla}\Phi + (\bar{E}\cdot\bar{n})\bar{\nabla}\Phi]\,da. \qquad 2.30$$

2.11 Since equation (2.30) holds only if $\bar{E}$ and $\bar{H}$ are continuous and have continuous first derivatives at all points of S, it cannot be directly applied to the problem of radiation from an aperture. To obtain the required extension, let us consider the surface S, closed at infinity, to be divided into two zones, $S_1$ and $S_2$, by a closed contour C on S as in Figure 2, page 12. $\bar{E}$ and $\bar{H}$ are continuous, have continuous first derivatives, and satisfy (2.02) to (2.05) over both $S_1$ and $S_2$. But the components of $\bar{E}$ and $\bar{H}$ which are tangential to the surface are subject to a discontinuous change on crossing C. This discontinuity can be reconciled with the field

$$E(\kappa;\lambda,\nu) = -\frac{i\mu}{\pi}\int_{\mathbb{R}^3} e^{i\kappa\cdot x}\left[(\underline{u}\times\underline{H})\hat{\Phi} + (\underline{u}\times\underline{E})\times\underline{\hat{A}}\hat{\Phi} + (\underline{E}\cdot\underline{u})\hat{A}\hat{\Phi}\right] dx .$$

$$E(\kappa;\lambda,\nu) = -\frac{i\mu}{\pi}\int_{\mathbb{R}^3} e^{i\kappa\cdot x}\left[(\underline{u}\times\underline{H})\hat{\Phi} + (\underline{u}\times\underline{E})\times\underline{\hat{A}}\hat{\Phi} + (\underline{E}\cdot\underline{u})\hat{A}\hat{\Phi}\right] dx .$$

equations by the further assumption of a line distribution of charges or currents about C. This line distribution contributes to the field and allows the resultant $\bar{E}$ and $\bar{H}$ vectors to satisfy the field equations.

2.12 Bottler (4) proposed a method of determining a suitable contour distribution. An abrupt change in the surface current

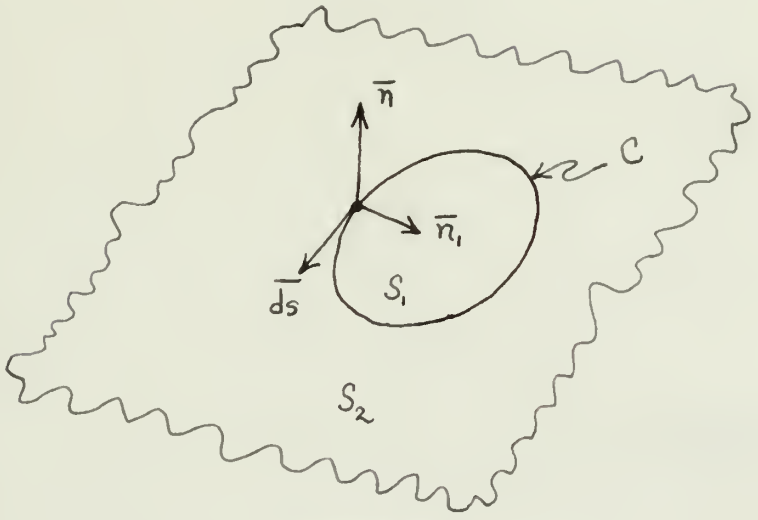
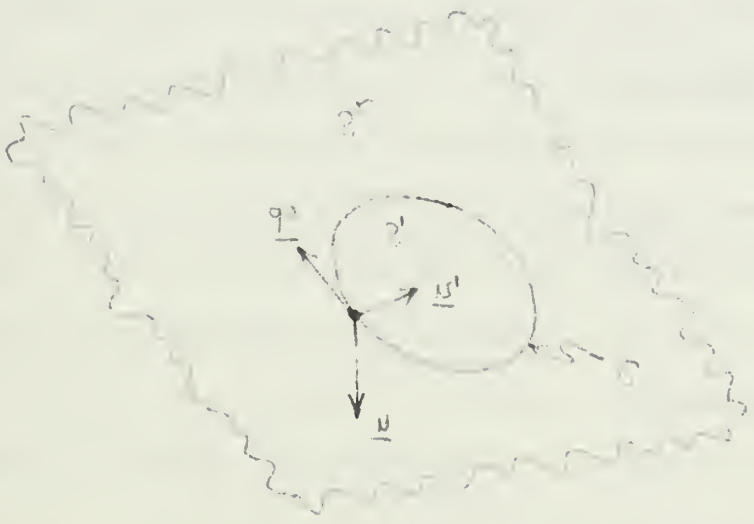



Figure 2

density is implied by the discontinuity the tangential components of $\bar{E}$ and $\bar{H}$ suffer on crossing C. The termination of a line of current, in turn, can be accounted for according to the equation of continuity by an accumulation of charge on C. Let $\bar{ds}$ be an element of length along C in the positive direction as determined by the positive $\bar{n}$ in Figure 2. $\bar{n}_1$ is a unit vector in the surface, normal to $\bar{ds}$ and $\bar{n}$, and directed into zone 1. The line densities of electric and magnetic charges will be designated by $\sigma$ and $\sigma^*$. Then the equations of continuity, (2.06) and (2.07), as applied to surface currents, become:

$$\bar{n}_1 \cdot (\bar{K}_1 - \bar{K}_2) = i\omega\sigma, \qquad 2.81$$



$$\bar{n}_1 \cdot (\bar{K}_1^* - \bar{K}_2^*) = i\omega\sigma^*, \qquad 2.32$$

where $\bar{K} \equiv$ electric current surface density,

$\bar{K}^* \equiv$ magnetic current surface density,

$\bar{K} = -\bar{n} \times \bar{H}$, $\bar{K}^* = \bar{n} \times \bar{E}$.

Expressing (2.31) and (2.32) in terms of $\bar{E}$ and $\bar{H}$,

$$i\omega\sigma = \bar{n}_1 \cdot (\bar{n} \times \bar{H}_2 - \bar{n} \times \bar{H}_1) = (\bar{H}_2 - \bar{H}_1) \cdot (\bar{n}_1 \times n), \qquad 2.33$$

$$i\omega\sigma^* = \bar{n}_1 \cdot (\bar{n} \times \bar{E}_1 - \bar{n} \times \bar{E}_2) = -(\bar{E}_2 - \bar{E}_1) \cdot (\bar{n}_1 \times n). \qquad 2.34$$

Now the electric field vector at a fixed point resulting from a surface charge distribution is

$$\bar{E}(X', Y', Z') = \frac{1}{4\pi\varepsilon} \int_S \rho(X, Y, Z)\, \bar{\nabla}\Phi \, da. \qquad 2.35$$

If S is considered as $S_1 - S_3$, where $C_3$ is a concentric circle inside C, and $S_3$ is allowed to approach $S_1$, in the limit $\rho(X, Y, Z)$ becomes $\sigma(X, Y, Z)$, da becomes ds, and the integral becomes completely defined by a contour integral. This contour integral describes the contribution of the charge accumulation on C to the electric field vector at the fixed observation point:

$$\bar{E}(X', Y', Z') = \bar{E}(R, \Theta) = \frac{1}{4\pi\varepsilon} \oint_C \sigma(a, \hat{\varphi})\, \bar{\nabla}\Phi \, ds. \qquad 2.36$$

From (2.33), $\sigma = \dfrac{(\bar{H}_2 - \bar{H}_1) \cdot (\bar{n}_1 \times \bar{n})}{i\omega}$.

As shown in Figure 2, $(\bar{n}_1 \times \bar{n}) = b\,(\bar{ds})$, where b is a constant such that $b\,(\bar{ds})$ is a unit vector. Over the surface $S_2$, the vectors $\bar{E}_2 = \bar{H}_2 = 0$. Therefore, $\sigma = -\dfrac{b}{i\omega}\,(\bar{H}_1 \cdot \bar{ds})$.

Substituting, (2.36) becomes:

$$\bar{\mu}' \cdot (\bar{K}'_{\times} - \bar{K}^{*}_{\times}) = \quad i\omega\sigma_{\times}$$

$$\bar{\mu} =$$

$$\bar{\mu} = - \bar{\mu}_{\times} \bar{\mu} = \bar{\mu}_{\times}$$

$$i\omega\sigma_{\times} = \bar{\mu}' \cdot (\bar{\mu} \times \bar{H}' - \bar{\mu} \times \bar{H}') = (\bar{H}'_{\times} - \bar{H}') \cdot (\bar{\mu}' \times \bar{\mu}) \ .$$

$$i\omega\sigma_{\times} = \bar{\mu}' \cdot (\bar{\mu} \times \bar{E}' - \bar{\mu} \times \bar{E}') = -(\bar{E}'_{\times} - \bar{E}') \cdot (\bar{\mu}' \times \bar{\mu}) \ .$$

$$E\,(X', \lambda', \xi') = \frac{1}{4\pi\varepsilon} \int \rho\,(X, \lambda, \xi)\, \Delta\Phi\ d\tau .$$

$$E\,(X', \lambda', \xi') = E(\kappa', \Theta) = \frac{1}{4\pi\varepsilon} \int \sigma(\nu', \psi)\, \Delta\Xi\ d\sigma \ .$$

$$\sigma = \frac{(\bar{H}'_{\times} - \bar{H}') \cdot (\bar{\mu}' \times \bar{\mu})}{i\omega} \ .$$

$$\sigma = - \frac{\bar{\mu}}{i\omega} \cdot (\ \bar{H}' \cdot \bar{q}\xi\ ) .$$

$$\bar{E}(R,\Theta) = -\frac{b}{i\omega\epsilon}\frac{1}{4\pi}\oint_C \bar{\nabla}\Phi\,(\bar{H}_1\cdot\bar{ds}), \qquad 2.37$$

which is the vector field occasioned by the discontinuity at C. Let $S_2$ represent an opaque screen over which $\bar{E}_2 = \bar{H}_2 = 0$, then from (2.30) and (2.37) may be written the equation for the field resulting from the radiating surface $S_1$ (restricting $0 < \beta < \pi$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ as shown in figure 3).

$$\bar{E}(R,\theta) = -\frac{b}{i\omega\epsilon}\frac{1}{4\pi}\oint_C \bar{\nabla}\Phi\,(\bar{H}_1\cdot\bar{ds}) - \frac{1}{4\pi}\int_S [i\omega\mu(\bar{n}\times\bar{H}_1)\Phi + (\bar{n}\times\bar{E}_1)\times\bar{\nabla}\Phi + (\bar{E}_1\cdot\bar{n})\bar{\nabla}\Phi\,]\,da. \qquad 2.38$$

Similarly for the magnetic field vector distribution, we write:

$$\bar{H}(R,\theta) = \frac{b}{i\omega\mu}\frac{1}{4\pi}\oint_C \bar{\nabla}\Phi\,(\bar{E}_1\cdot\bar{ds}) + \frac{1}{4\pi}\int_S [i\omega\epsilon(\bar{n}\times\bar{E}_1)\Phi - (\bar{n}\times\bar{H}_1)\times\bar{\nabla}\Phi - (\bar{n}\cdot\bar{H}_1)\bar{\nabla}\Phi\,]\,da. \qquad 2.39$$

2.13 The further development of equations (2.38) and (2.39) will for the present be confined to circularly symmetric, plane polarized $\bar{E}$ and $\bar{H}$ distributions. Since only $\bar{E}_1$ and $\bar{H}_1$ exist, they will be written as $\bar{E}$ and $\bar{H}$. Let

$$\bar{E} = I(\hat{\rho}, a)\,\bar{i}, \qquad 2.40$$

$$\bar{H} = M(\hat{\rho}, a)\,\bar{j}, \qquad 2.41$$

and the aperture is in the (X,Y,0) plane so $\bar{n} \equiv \bar{k}$. (Figure 3)*

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* For electrically polarized plane waves, the $\bar{E}$ and $\bar{H}$ vectors have directions independent of time and are defined by:

$$|H| = \frac{|E|}{\eta}, \quad \eta = \sqrt{\frac{\mu}{\epsilon}}, \quad c = \frac{1}{\sqrt{\mu\epsilon}} = f\lambda, \quad k = \frac{2\pi}{\lambda}.$$

The coefficients in the above expressions have the following approximate free space values:

$\eta_o = 120\pi$ ohms, $\mu_o = 1.257\times10^{-6}$ henries/meter,

$c_o = 3\times10^{8}$ meters/second, $\epsilon_o = 8.854\times10^{-12}$ farads/meter.

$$E(\theta,\theta') = -\frac{ik}{4\pi}\int_{\Gamma} \Phi\,\Delta\Phi'\,(\bar{H}'\cdot d\bar{s}),$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$E(\theta,\theta') = -\frac{ik}{4\pi}\int_{\Gamma} \Phi\,\Delta\Phi'\,(\bar{H}'\cdot d\bar{s})$$

$$-\frac{4\pi}{ik}\int_{\Gamma}\left[ik(\bar{\mu}\times\bar{H}')\Phi + (\bar{\mu}\times\bar{E}')\times\bar{\nabla}\Phi + (\bar{E}'\cdot\bar{\mu})\bar{\nabla}\Phi\right]\cdot d\bar{\sigma}.$$

$$H(\theta,\theta') = \frac{ik}{4\pi}\int_{\Gamma} \Phi\,\Delta\Phi'\,(\bar{E}'\cdot d\bar{s})$$

$$+\frac{4\pi}{ik}\int_{\Gamma}\left[ik(\bar{\mu}\times\bar{E}')\Phi - (\bar{\mu}\times\bar{H}')\times\bar{\nabla}\Phi - (\bar{\mu}\cdot\bar{H}')\bar{\nabla}\Phi\right]\cdot d\bar{\sigma}.$$

$$\bar{H} = H(\theta,\phi)\,\hat{\theta}\,,$$

$$\bar{E} = E(\theta,\phi)\,\hat{\theta}\,.$$

$$|H| = \frac{J}{|E|}\,,\quad J = \sqrt{\frac{\varepsilon}{\mu}}\,,\quad C = \frac{1}{\sqrt{\mu\varepsilon}} = 1\gamma\,,\quad k = \frac{\nu}{\sqrt{\mu}}\,.$$

$$\bar{\mu}^{o} = \qquad\qquad \bar{H}^{o} = \qquad\qquad -\zeta$$

$$\bar{C}^{s} = \qquad\qquad \bar{C}^{o} = \qquad\qquad -\zeta\gamma$$

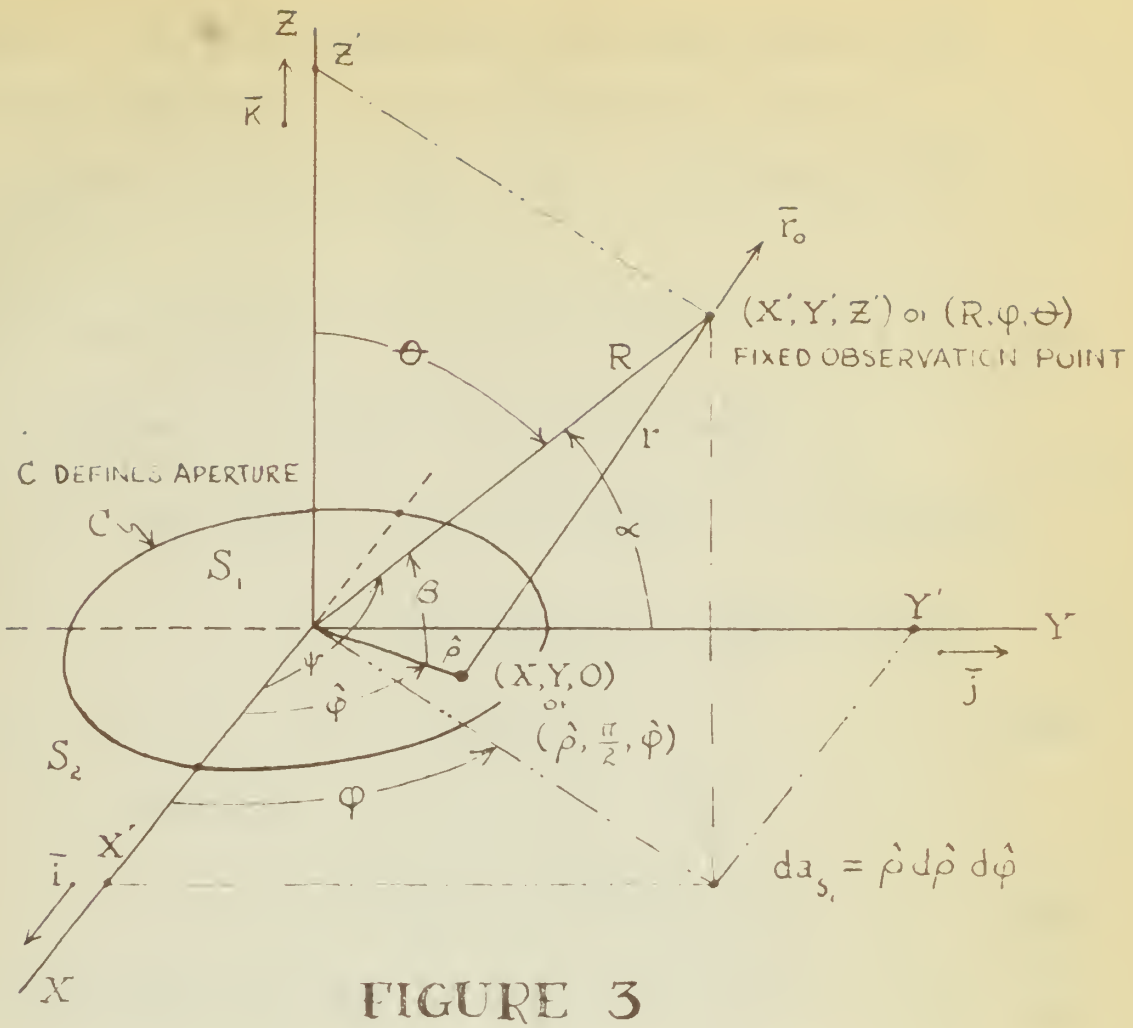


FIGURE 3

$R_Y = R\cos\alpha = R\sin\Theta\sin\varphi,$

$R_X = R\cos\psi = R\sin\Theta\cos\varphi,$

$R_Z = R\cos\Theta,$

$R\cos\beta = R\sin\Theta\cos(\hat{\varphi} - \psi).$

$\therefore \cos\beta = \sin\Theta\cos(\hat{\varphi} - \varphi);\ \cos\psi = \sin\Theta\cos\varphi;\ \cos\alpha = \sin\Theta\sin\varphi.$

$$r = \sqrt{R^2 + \hat{\rho}^2 - 2R\hat{\rho}\cos\beta} = R\sqrt{1 - \frac{2\hat{\rho}}{R}\cos\beta + \frac{\hat{\rho}^2}{R^2}}\ .$$

For $R >>> \hat{\rho}$,

$$r = R\left(1 - \frac{2\hat{\rho}}{R}\cos\beta\right)^{\frac{1}{2}} = R\left(1 - \frac{2\hat{\rho}}{2R}\cos\beta + \cdots\right)$$

$$= R - \hat{\rho}\cos\beta = R - \hat{\rho}\sin\Theta\cos(\hat{\varphi} - \varphi).$$

2.14 The first integral in (2.38) vanishes because of the circular symmetry of the aperture distribution, ( $\bar{H}_1 \cdot \overline{ds}$ ) cancelling in pairs around the contour C. Therefore, we write (2.38) as:

$$4\pi \bar{E}(R,\theta) = -\int_S [i\omega\mu(\bar{n}\times\bar{H})\Phi + (\bar{n}\times\bar{E})\times\bar{\nabla}\Phi + (\bar{n}\cdot\bar{E})\bar{\nabla}\Phi]da. \quad 2.42$$

Investigating the components of (2.42), we observe

$$a.\ i\omega\mu(\bar{n}\times\bar{H})\Phi = i\omega\mu[\bar{\kappa}\times M(\hat{\rho},a)\bar{j}] = -i\omega\mu M(\hat{\rho},a)\bar{i}$$

$$= -\frac{i\omega\mu}{\eta} I(\hat{\rho},a)\bar{i} = -i\omega\sqrt{\mu\varepsilon}\ I(\hat{\rho},a)\bar{i}$$

$$= -ik\, I(\hat{\rho},a)\bar{i}, \quad 2.43a$$

$$b.\ (\bar{n}\times\bar{E})\times\bar{\nabla}\Phi = (\bar{\kappa}\times I(\hat{\rho},a)\bar{i}\,)\times\bar{\nabla}\Phi$$

$$= I(\hat{\rho},a)\,[\bar{j}\times\bar{\nabla}\Phi], \quad 2.43b$$

$$c.\ \bar{\nabla}\Phi = -ik\Phi\left(1+\frac{1}{ikr}\right)\bar{r}_o, \quad 2.43c$$

$$d.\ (\bar{n}\times\bar{E})\times\bar{\nabla}\Phi = -ik\Phi I(\hat{\rho},a)\left[1+\frac{1}{ikr}\right][\bar{j}\times\bar{r}_o]. \quad 2.43d$$

The following geometrical relations are obtained from Figure 5 where $r \gg\gg \hat{\rho}$ :

$$\bar{r}_o = \bar{i}\sin\theta\cos\varphi + \bar{j}\sin\theta\sin\varphi + \bar{\kappa}\cos\theta,$$

$$\therefore\ \bar{j}\times\bar{r}_o = \bar{i}\cos\theta - \bar{\kappa}\sin\theta\cos\varphi,$$

and therefore,

$$(\bar{n}\times\bar{E})\times\bar{\nabla}\Phi = -ik\Phi I(\hat{\rho},a)\left[1+\frac{1}{ikr}\right][\bar{i}\cos\theta - \bar{\kappa}\sin\theta\cos\varphi], \quad 2.44$$

$$(\bar{n}\cdot\bar{E})\bar{\nabla}\Phi = I(\hat{\rho},a)\,[\bar{\kappa}\cdot\bar{i}\,]\,\bar{\nabla}\Phi = 0. \quad 2.45$$

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$$(\hat{u}\cdot\vec{E})\vec{A}^{\vec{E}} = I(\hat{y}/\sigma)\,\Big[\,K\cdot\,\Big]\,\vec{A}^{\vec{E}} = 0\,.$$

$$(\hat{u}\times\vec{E})\times\vec{A}^{\vec{E}} = -ik^{\vec{E}}I(\hat{y}/\sigma)\Big[1+\frac{ik\sigma}{2}\Big]\Big[i\cos\theta - \kappa\,\sin\theta\cos\phi\Big]\,,$$

$$\therefore\ \Big[\,\times\vec{t}^{\,\vec{E}} = \quad i\cos\theta - \kappa\,\sin\theta\cos\phi\,,$$

$$\vec{t}^{\,\vec{E}} = i\,\sin\theta\cos\phi + \Big[\,\sin\theta\sin\phi + \quad \kappa\cos\theta\,,$$

$$\gg\hat{y}\,:$$

$$(\hat{u}\times\vec{E})\times\vec{A}^{\vec{E}} = -ik^{\vec{E}}\,I(\hat{y}/\sigma)\Big[1+\frac{ik\sigma}{2}\Big]\Big[\frac{1}{ik\sigma}\Big]\Big[\,\times\vec{t}^{\,\vec{E}}\Big]\,.$$

$$\vec{A}^{\vec{E}} = -ik^{\vec{E}}\,\left(1+\frac{ik\sigma}{1}\right)\vec{E}^{\,\vec{E}}\,,$$

$$= I(\hat{y}/\sigma)\,\Big[\,\times\vec{A}^{\vec{E}}\Big]\,,$$

$$(\hat{u}\times\vec{E})\times\vec{A}^{\vec{E}} = (\vec{E}\times I(\hat{y}/\sigma)\vec{t}\,)\times\vec{A}^{\vec{E}}$$

$$= -ik\,I(\hat{y}/\sigma)\vec{t}^{\,\vec{E}}\,,$$

$$= \frac{-ik\sigma}{1}\,I(\hat{y}/\sigma)\vec{t} \quad = -ik\sigma\vec{E}\quad I(\hat{y}/\sigma)\vec{t}$$

$$ik\sigma\,(\hat{u}\times\vec{H})^{\vec{E}} = ik\sigma\Big[\vec{E}\times H(\hat{y}/\sigma)\vec{t}\,\Big] = -ik\sigma\,H\,(\hat{y}/\sigma)\vec{t}$$

$$4\pi\vec{E}(\hat{u}/\sigma) = -\int_{\sigma}\Big[\,ik\sigma(\hat{u}\times\vec{H})^{\vec{E}} + (\hat{u}\times\vec{E})\times\vec{\nabla}^{\vec{E}} + (\hat{u}\cdot\vec{E})\,\vec{\nabla}^{\vec{E}}\Big]_{\sigma}\,d\sigma\,.$$

Substituting (2.43), (2.44), (2.45) in equation (2.42), we obtain:

$$4\pi\bar{E}(R,\theta) = ik\int_S \Phi I(\hat{\rho},a)\left[\bar{i}+\left(1+\frac{1}{ikr}\right)(\bar{i}\cos\theta-\bar{k}\sin\theta\cos\varphi)\right]da. \qquad 2.46$$

For the distant field where $r \ggg 1$, $\left(1+\frac{1}{ikr}\right)\to 1$.

Therefore (2.46) can be rewritten:

$$4\pi\bar{E}(R,\theta) = ik\left[(1+\cos\theta)\bar{i}-(\sin\theta\cos\varphi)\bar{k}\right]\int_S \Phi I(\hat{\rho},a)\,da. \qquad 2.47$$

and remembering that $\Phi = \frac{e^{-ikr}}{r}$,

$$4\pi\bar{E}(R,\theta) = \frac{2\pi i}{\lambda}\left[(1+\cos\theta)\bar{i}-(\sin\theta\cos\varphi)\bar{k}\right]\int_S \frac{e^{-ikr}}{r} I(\hat{\rho},a)\,da. \qquad 2.48$$

From the geometry of Figure 3

$$da = \hat{\rho}\,d\hat{\rho}\,d\hat{\varphi}$$

$r = R - \hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)$, {For the distant field $R \ggg \hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)$}, therefore let $r = R$.

Substituting these geometrical relations in (2.48), we obtain:

$$\bar{E}(R,\theta) = \frac{2\pi i}{4\pi\lambda}\left\{\left[1+\cos\theta\right]\bar{i} - \left[\sin\theta\cos\varphi\right]\bar{k}\right\}\left\{\frac{e^{-ikR}}{R}\right\}\int_S I(\hat{\rho},a)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\hat{\rho}\,d\hat{\rho}\,d\hat{\varphi},$$

$$= \frac{\bar{G}}{2\pi}\int_{\hat{\rho}=0}^{a}\int_{\hat{\varphi}=0}^{2\pi} I(\hat{\rho},a)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\varphi}-\varphi)}\hat{\rho}\,d\hat{\rho}\,d\hat{\varphi}, \qquad 2.49$$

where $\bar{G} = \frac{i\pi e}{\lambda R}^{i(\omega t-kR)}\left[(1+\cos\theta)\bar{i}-(\sin\theta\cos\varphi)\bar{k}\right]$, $\qquad 2.50$

in which the time factor $e^{i\omega t}$ has been included.

2.16 Since $\varphi$ is independent of $\hat{\varphi}$, $d\hat{\varphi} = d(\hat{\varphi}-\varphi)$, and letting $k\hat{\rho}\sin\theta = w$, equation (2.49) becomes

$$\bar{E}(R,\theta) = \frac{\bar{G}}{2\pi}\int_{\hat{\rho}=0}^{a} I(\hat{\rho},a)\hat{\rho}\,d\hat{\rho}\int_{\hat{\varphi}=0}^{2\pi} e^{iw\cos(\hat{\varphi}-\varphi)}\,d(\hat{\varphi}-\varphi). \qquad 2.51$$

$$\bar{E}(R,\theta) = \frac{\gamma}{\epsilon}\int_{\hat{\phi}=0}^{2\pi}\int_{\hat{\rho}=0}^{a} I(\hat{\rho},\alpha)\,\hat{\rho}\,d\hat{\rho}\,e^{ik\cos(\hat{\phi}-\phi)}\,d(\hat{\phi}-\phi).$$

$$\bar{E} = \frac{\gamma k}{i\omega\epsilon}\,e^{i(\omega t - kR)}\left[(1+\cos\theta)\bar{i} - (\sin\theta\cos\phi)\bar{k}\right].$$

$$= \frac{\gamma k}{\epsilon}\int_{\hat{\rho}=0}^{a}\int_{\hat{\phi}=0}^{2\pi} I(\hat{\rho},\alpha)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\phi}-\phi)}\,\hat{\rho}\,d\hat{\rho}\,d\hat{\phi}$$

$$-[\sin\theta\cos\phi]\bar{k}\Big\}\Big\{\frac{k}{e^{-ikR}}\Big\}\int_{\hat{\rho}=0}^{a}\int_{\hat{\phi}=0}^{2\pi} I(\hat{\rho},\alpha)\,e^{ik\hat{\rho}\sin\theta\cos(\hat{\phi}-\phi)}\,\hat{\rho}\,d\hat{\rho}\,d\hat{\phi}\,,$$

$$E(R,\theta) = \frac{4\pi\gamma}{2\pi i}\Big\{[1+\cos\theta]\bar{i}$$

$$\gg\gg\ \hat{\rho}\quad\theta\quad\hat{\phi}\text{-}\phi\ \Big\} =$$

$$= -\hat{\rho}\ \theta\quad\hat{\phi}\text{-}\phi$$

$$= \hat{\rho}\,d\hat{\rho}\,d\hat{\phi}$$

$$4\pi E(R,\theta) = \frac{\gamma}{2\pi i}\Big[(1+\cos\theta)\,\bar{i} - (\sin\theta\cos\phi)\,\bar{k}\Big]\Big\{\frac{k}{2}\Big\}\frac{e^{-ikr}}{r}\int I(\hat{\rho},\alpha)\,d\alpha.$$

$$\bar{A} = \frac{k}{c}\,\frac{e^{-ikr}}{r}\,.$$

$$4\pi E(R,\theta) = ik\Big[(1+\cos\theta)\,\bar{i} - (\sin\theta\cos\phi)\bar{k}\Big]\Big\{\frac{k}{2}\Big\}\int \bar{\phi}\,I(\hat{\rho},\alpha)\,d\alpha.$$

$$\gg\gg\quad (1+\frac{1}{ikr}) \to 1.$$

$$-\bar{i}\,\Big\{\frac{k}{2}\Big\} = ik\Big\}\bar{\phi}\,I(\hat{\rho},\alpha)\Big[\bar{i} + (1+\frac{1}{ikr})(\bar{i}\cos\theta - \bar{k}\sin\theta\cos\phi)\Big]\,d\alpha\,.$$

From Jahnke and Emde (6), page 149,

$$J_n(w) = \frac{i^{-n}}{2\pi} \int_{\varphi=0}^{2\pi} e^{iw\cos\varphi}\, e^{in\varphi}\, d\varphi,$$

and in the case of $n = 0$,

$$J_0(w) = \frac{1}{2\pi} \int_{\varphi=0}^{2\pi} e^{iw\cos\varphi}\, d\varphi. \qquad 2.52$$

Substituting (2.52) into (2.51), we obtain:

$$\bar{E}(R,\theta) = \bar{G} \int_{\hat{\rho}=0}^{a} \hat{\rho}\, I(\hat{\rho},a)\, J_0(k\hat{\rho}\sin\theta)\, d\hat{\rho}. \qquad 2.53$$

Now let $z = \frac{\hat{\rho}}{a}$, $\hat{\rho} = 0$, $dz = \frac{d\hat{\rho}}{a}$, and for circularly symmetric aperture distributions,

$$\bar{E}(R,\theta) = \bar{G}a^2 \int_{z=0}^{1} z\, I(z)\, J_0(zka\sin\theta)\, dz. \qquad 2.54$$

This is the form which will be considered for further development of a solution.

2.17 In completely parallel fashion from (2.32),

$$\bar{H}(R,\theta) = \bar{F} \int_{\hat{\rho}=0}^{a} M(\hat{\rho},a)\hat{\rho}\, J_0(k\hat{\rho}\sin\theta)\, d\hat{\rho}, \qquad 2.55$$

where $$\bar{F} = \frac{ire^{i(\omega t - kR)}}{\lambda R}\left[(1+\cos\theta)\,\bar{j} - (\sin\theta\sin\varphi)\,\bar{k}\right], \qquad 2.56$$

and letting $z = \frac{\hat{\rho}}{a}$,

$$\bar{H}(R,\theta) = \bar{F}a^2 \int_{z=0}^{1} z\, M(z)\, J_0(zka\sin\theta)\, dz. \qquad 2.57$$

2.18 The mean value of the rate at which energy traverses a surface whose normal is in the direction $(\bar{E}\times\bar{H})$ in an electromagnetic field can be determined by constructing the complex Poynting vector

$$\bar{S}^* = \tfrac{1}{2}\,\bar{E}\times\tilde{\bar{H}}, \qquad 2.58$$

where $\tilde{\bar{H}}$ is the complex conjugate of $\bar{H}$. This corresponds to the measurable energy in practice. Substituting (2.54) and (2.57) in

$$\bar{S}_{av} = \frac{1}{2} \bar{E} \times \bar{H}^{*},$$

$$\left( \quad \right)$$

$$\bar{H}(\kappa,\theta) = \bar{E}_{\alpha} \int_{\xi=0}^{} N(\xi)\, J_0^{'}(\xi k a \sin\theta)\, d\xi .$$

$$= \frac{\hat{\phi}}{\eta} ,$$

$$\bar{E} = \frac{jk\epsilon}{4\pi} \frac{e^{j(\omega t - kR)}}{R} \left[ (1+\cos\theta) \right] - (\sin\theta \sin\phi)\, \bar{\kappa} \Big] ,$$

$$\bar{H}(\kappa,\theta) = \bar{E} \int_{\hat{\rho}=0}^{a} W(\hat{\rho}_{\alpha})\, \hat{\rho}\, J_0^{'}(k\hat{\rho}_{\alpha} \sin\theta)\, d\hat{\rho}_{\alpha} ,$$

$$\bar{E}(\kappa,\theta) = \sum_{\xi=0}^{} \bar{E}_{\alpha} \int N(\xi)\, J_0^{'}(\xi k a \sin\theta)\, d\xi .$$

$$= \frac{\hat{\phi}}{\eta}$$

$$\bar{E}(\kappa,\theta) = \sum_{\hat{\rho}=0}^{a} \int_{\hat{\rho}=0}^{} I(\hat{\rho}_{\alpha})\, J_0^{'}(k\hat{\rho}_{\alpha} \sin\theta)\, d\hat{\rho}_{\alpha} .$$

$$J_0^{'}(x) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} e^{\,j x \cos\phi}\, d\phi .$$

$$= \frac{1}{2\pi} \int_{\phi=0}^{2\pi} e^{\,j(x\cos\phi - n\phi)}\, d\phi ,$$

$$J_n(x) = \frac{1}{2\pi} \int_{\phi=0}^{2\pi} e^{\,j(x\cos\phi - n\phi)}\, d\phi ,$$

(2.58), we obtain:

$$\bar{S}^* = \frac{\bar{E}}{2} \times \tilde{\bar{H}}$$

$$= \left[\frac{a^4}{2}\left\{\int_0^1 z I(z) J_0(zka\sin\theta)dz\right\}\left\{\int_0^1 z M(z) J_0(z ka\sin\theta)dz\right\}\right](\bar{G}\times\tilde{\bar{F}}). \qquad 2.59$$

If $\theta$ is small*, (2.58) becomes

$$\bar{G} = \frac{i\pi e}{\lambda R}^{i(\omega t - kR)}\left[1+\cos\theta\right]\bar{i}, \qquad 2.60$$

and (2.56) becomes

$$\bar{F} = \frac{i\pi e}{\lambda R}^{i(\omega t - kR)}\left[1+\cos\theta\right]\bar{j}. \qquad 2.61$$

Now (2.59) may be written

$$\bar{S}^* = \frac{\pi^2 a^4}{2\lambda^2 R^2}\left[\int_0^1 z I(z) J_0(zka\sin\theta)dz\right]\left[\int_0^1 z M(z) J_0(zka\sin\theta)dz\right]\left[1+\cos\theta\right]^2\bar{k}. \qquad 2.62$$

Also when $\theta$ is small

$$|E| = \left|\frac{\pi a^2}{\lambda R}(1+\cos\theta)\left[\int_0^1 z I(z) J_0(zka\sin\theta)\,dz\right]\right|, \qquad 2.63$$

$$|H| = \left|\frac{\pi a^2}{\lambda R}(1+\cos\theta)\left[\int_0^1 z M(z) J_0(zka\sin\theta)\,dz\right]\right|, \qquad 2.64$$

and since $|H| = \frac{|E|}{\eta}$ for polarized plane waves, (2.62) may be written:

$$|\bar{S}^*| = \left|\frac{\pi^2 a^4}{2\lambda^2 R^2\eta}\left[\left\{\int_0^1 z I(z) J_0(zka\sin\theta)\,dz\right\}\left\{1+\cos\theta\right\}\right]^2\right|. \qquad 2.65$$

---

* The restriction that $\theta$ be small is almost a universal one. Investigation of this problem with the restriction on $\theta$ removed would be very interesting and will be considered if time permits. Paragraph (2.12) applies the restriction $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ which cannot be removed using the present method of solution.

$$\frac{1}{2}\int_{S} \vec{E} \times \vec{H}^{*} \cdot \hat{n}\, dS$$

$$= \left[\frac{\pi}{4}\right] \int_{0}^{a} J_1(\xi) J_0^2(\xi a \sin\theta)\, d\xi \int_{0}^{a} \xi\, N(\xi) J_0^2(\xi k \sin\theta)\, d\xi \left[\hat{z} \times \hat{E}\right].$$

$$\vec{E} = \frac{iW\epsilon}{\gamma k} e^{i(\omega t - kr)} \left[1 + \cos\theta\right] \hat{i}.$$

$$\vec{S}_{av} = \frac{\gamma k}{2W\epsilon} \left[\int_{0}^{a} J_1(\xi) J_0^2(\xi a \sin\theta)\, d\xi\right]^2 \int_{0}^{a} N(\xi) J_0^2(\xi k \sin\theta)\, d\xi \left[1 + \cos\theta\right] \hat{k}.$$

$$|E| = \left| \frac{\gamma k}{W\epsilon} (1 + \cos\theta) \int_{0}^{a} \xi\, J_0^2(\xi k \sin\theta)\, d\xi \right|,$$

$$|H| = \left| \frac{\gamma k}{W\epsilon} (1 + \cos\theta) \int_{0}^{a} N(\xi) J_0^2(\xi k \sin\theta)\, d\xi \right|.$$

$$\frac{|H|}{|E|} = \frac{J}{|E|}$$

$$|I_{ak}| = \left| \frac{2\gamma\epsilon k J}{W\epsilon^2} \int_{0}^{a} J_1(\xi)\, J_0(\xi k \sin\theta)\, d\xi \right| \left\{1 + \cos\theta\right\}.$$

Thus from (2.65) it is seen that for polarized plane waves $|S^*| = \frac{|E^2|}{2\eta} = \frac{|H^2|}{2}\eta$ or the measured energy is proportional to the square of $|S|$ or $|H|$. This will be used as the correlating relationship between the analytic and practical factors in the problem.

## Chapter III

### A Method for Determination of Aperture Distribution

3.1 Chapter II dealt with the development of the integral equation relationship between the distant field vectors $\bar{E}$ and $\bar{H}$ and the aperture distributions $I(z)$ and $M(z)$, restricted to circularly symmetric plane polarized waves. Since the measurable energy rate in the distant field, $|S^*|$ is proportional to $|E^2|$, only equation (2.54) will be considered.

$$\bar{E}(R,\theta) = \bar{G}a^2 \int_0^1 z\, I(z) J_0(zka\sin\theta)\, dz \qquad 3.01$$

3.2 One possible amplitude distribution at the aperture is $I(z) = I$, a constant. This distribution will be used to determine the functional dependence of $\bar{E}$ in order to bring equation (3.01) into usable form.

$$\bar{E}(R,\theta) = I\bar{G}a^2 \int_0^1 z\, J_0(zka\sin\theta)\, dz \qquad 3.02$$

To integrate (3.02) we can use Sonine's first finite integral*

$$\int_0^{\frac{\pi}{2}} J_\mu(z'\sin\theta') \sin^{\mu+1}\theta' \cos^{2\nu+1}\theta' d\theta' = \tfrac{1}{2}\Gamma(\nu+1)\frac{J_{\mu+\nu+1}(z')}{\left(\frac{z'}{2}\right)^{\nu+1}}, \qquad 3.03$$

$$[\Re(\mu) > -1;\ \Re(\nu) > -1]$$

and make the following substitutions:

$$z = \sin\theta', \qquad dz = \cos\theta' d\theta',$$
$$\mu = 0, \qquad ka\sin\theta = z'.$$
$$\nu = 0,$$

Sonine's integral (3.03) then becomes:

$$\int_0^1 J_0(zka\sin\theta)\, z\, dz = \tfrac{1}{2}\Gamma(1)\frac{J_1(ka\sin\theta)}{\left(\frac{ka\sin\theta}{2}\right)}. \qquad 3.04$$

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* Theory of Bessel Functions, Watson, G.N., (5), Page 373

$$\int_0^1 J_\nu(\alpha x)\sin\theta)\, x\, dx = \frac{1}{\alpha^2} L_\nu(\alpha) \frac{\left(\frac{\alpha}{2}\sin\theta\right)^\nu}{J_\nu(\alpha\sin\theta)} \cdot$$

$$\lambda = C^2$$
$$\gamma r = C^2 \qquad \theta = r'$$
$$r = \theta' \qquad r = \theta' dr'$$

$$\left[\Gamma(n)\right)^{-1}; \Gamma(n)\rangle -1\Big]$$

$$\int_0^{\frac{\pi}{2}} J_\nu(x\sin\theta)^{\nu} \sin^{2\mu+1}\theta\, d\theta = \frac{\sqrt{\pi}}{2} L(\nu+1) \frac{\left(\frac{x}{2}\right)^{\nu+1}}{\Gamma_{\mu+\nu+1}(x')} ;$$

$$E(k|\theta) = \int_0^{\theta} \left[1 - \frac{1}{2}k^2 \sin^2\theta\right)^{\frac{1}{2}} d\theta$$

$$= \int_0^{\theta} \left[1 - \frac{1}{2}k^2 I(x) J_0^2(x\sin\theta)\right) dx$$

$$E(k|\theta) = \int_0^{\theta} \left[1 - \frac{1}{2}k^2 \sin^2\theta\right)^{\frac{1}{2}} d\theta$$

Therefore, (3.02) becomes:

$$\bar{E}(R,\theta) = I\bar{G}a^2\Gamma(1)\frac{J_1(ka\sin\theta)}{(ka\sin\theta)}, \qquad 3.05$$

which shows that the contribution of the integral to $\bar{E}(R,\theta)$ is a function of $(ka\sin\theta)$. Therefore, the substitution $(ka\sin\theta = x$ will be made; and $\bar{E}(R,\theta)$ will be described as $\bar{E}(x)$, remembering, however, that it also depends on R and $\theta$ as they occur in $\bar{G}$. Thus the general equation to be used for circularly symmetric aperture distribution becomes:

$$\bar{E}(x) = \bar{G}a^2\int_0^1 z\,I(z)\,J_0(xz)\,dz. \qquad 3.06$$

3.3 How the sample integration of (3.02) by the use of (3.05) gives rise to a practical method of solution. For by the very character of Sommerfeld's formula (3.02), it is seen that the formula will work equally well whenever $I(z) = I(1-z^2)^n$. With $z = \sin\theta'$, $(1-z^2)$ becomes $\cos^2\theta'$ and merely gives a new value of $\nu$ in (3.03), x and $\mu$ retaining values originally assigned. Silver (8) and Spencer (9, 10) have previously recognized the value of this integration restricted to the single term $(1-z^2)^n$.

3.4 The problem immediately at hand is then the expression of a circularly symmetric aperture distribution as a function of $(1-z^2)$. Assume, furthermore, that the distribution can be expressed as follows:

$$I(z) = \sum_{n=0}^{\infty} I_{\frac{n+1}{2}}\,(1-z^2)^{\frac{n-1}{2}} \qquad 3.07$$

$$= \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} - z^2\sum_{n=0}^{\infty}\left(\frac{n+1}{2}\right)\frac{I_{\frac{n+1}{2}}}{1!} + z^4\sum_{n=0}^{\infty}\left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}\right)\frac{I_{\frac{n+1}{2}}}{2!} + \cdots . \qquad 3.08$$

$$= \sum_{\nu=0}^{\infty} I_{\nu+1} - \frac{s}{1!}\sum_{\mu=0}^{\infty}\left(\frac{s}{2}\right)I_{\mu+1} + \frac{s}{2!}\sum_{\mu=0}^{\infty}\left(\frac{s}{2}\right)\left(\frac{s}{2}-1\right)I_{\mu+1} + \cdots .$$

$$I(s) = \sum_{\nu=0}^{\infty} I_{\nu+1} \quad (1-s)^{\nu}$$

$$E(x) = \int_0^{\infty} e^{-xs}\, I(s)\, J_0(xs)\, ds .$$

$$E(k,\phi) = \frac{1}{(k\sin\theta)} \sum_{\nu=0}^{\infty} q_{\nu+1}(1)\, T_{\nu}(k\sin\theta) .$$

It is helpful to note that whenever $\frac{n-1}{2}$ is an integer, $(1-z^2)^{\frac{n-1}{2}}$ can be expressed in a finite number of terms, but that a fractional exponent (which has been shown necessary in some cases, Appendix III, example d) requires an infinite series. There are at present no known series expressions for $\bar{E}(x)$ involving fractional exponents, but fractional exponents are necessary in the expression for $I(z)$ to include all possibilities. If the summation

$$I(z) = \sum_{n=0}^{\infty} I_{n+1} (1-z^2)^n \qquad 3.09$$

is considered, then

$$I(z) = \sum_{n=0}^{\infty} I_{n+1} - z^2 \sum_{n=1}^{\infty} n I_{n+1} + z^4 \sum_{n=2}^{\infty} \frac{n(n-1)}{2!} I_{n+1} - z^6 \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{3!} I_{n+1} + \cdots . \qquad 3.10$$

3.5 With $I(z)$ expressed as in (3.07), equation (3.06) becomes:

$$\bar{E}(x) = \bar{G} a^2 \int_0^1 z J_0(xz) \left[ \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} (1-z^2)^{\frac{n-1}{2}} \right] dz . \qquad 3.11$$

Interchanging the order of summation and integration, we have

$$\bar{E}(x) = \bar{G} a^2 \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} \int_0^1 z J_0(xz) (1-z^2)^{\frac{n-1}{2}} dz . \qquad 3.12$$

Now making the substitution suggested by paragraph (3.5), $z = \sin\theta'$, the integral in (3.12) becomes:

$$\int_0^{\frac{\pi}{2}} J_0(x \sin\theta') \sin\theta' \cos^n\theta' \, d\theta' , \qquad 3.13$$

which is recognized as the integral in (3.08) with $\mu = 0$, and $\nu = \frac{n-1}{2}$. Therefore, using (3.08) and (3.13), (3.12) becomes:

$$\bar{E}(x) = \frac{\bar{G} a^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} \Gamma\left(\frac{n+1}{2}\right) \frac{J_{\frac{n+1}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{n+1}{2}}} . \qquad 3.14$$

Formula (3.14) is the basis of the solution of this practical method.

3.6 It is important now to investigate (3.14). First, the

$$\Gamma(\epsilon) = \sum_{n=0}^{\infty} \Gamma^{n+1} (1-\epsilon)^n$$

$$\Gamma(\epsilon) = \sum_{n=0}^{\infty} \Gamma^{n+1} - \epsilon \sum_{n=1}^{\infty} n \Gamma^{n+1} + \epsilon^2 \sum_{n=2}^{\infty} \frac{n(n-1)}{2!} \Gamma^{n+1} - \epsilon^3 \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)}{3!} \Gamma^{n+1} + \cdots$$

$$\Sigma(x) = \int_0^{\infty} \epsilon^{x-1} \left[ \sum_{n=0}^{\infty} \Gamma^{n+1} (1-\epsilon)^n \right] d\epsilon .$$

$$\Sigma(x) = \sum_{n=0}^{\infty} \Gamma^{n+1} \int_0^{\infty} \epsilon^{(x+n)-1} (1-\epsilon)^n \, d\epsilon .$$

$$= \int_0^{\infty} \zeta^{(x+n)-1} e^{-\zeta} \, d\zeta ,$$

$$\lambda = \frac{\Gamma}{1+\Gamma}$$

$$\Sigma(x) = \frac{1}{(1+\Gamma)^x} \sum_{n=0}^{\infty} \Gamma^{n+1} \Gamma(x+n) \frac{1}{n!} \frac{1}{(1+\Gamma)^n}$$

function*, as defined by Jahnke and Emde (6), will be introduced.

$$\Lambda_n(x) = \frac{n!}{\left(\frac{x}{2}\right)^n} J_n(x), \qquad 3.15$$

and

$$\Lambda_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \frac{n!}{(n+k)!} \left(\frac{x}{2}\right)^{2k}. \qquad 3.16$$

In both (3.15) and (3.16), $n = 0,1,2,3, \ldots$ , but for the purposes of this work an extension of the definition of $\Lambda_n$ to half-integer values of n is necessary. Formula (3.16) is obtained directly from the series definition of $J_n(x)$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!} \left(\frac{x}{2}\right)^{n+2k}. \qquad 3.17$$

Now (3.17) can be extended to the case for n not an integer** by employing Gauss's function $\Pi(p)$ instead of $p!$. In the case p is an integer, $p! = \Pi(p) = \Gamma(p+1)$, and for all p, $\Pi(p) = \Gamma(p+1)$. Therefore, it is a logical extension to define

$$\Lambda_{\frac{n+1}{2}}(x) = \Pi\left(\frac{n+1}{2}\right) \frac{J_{\frac{n+1}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{n+1}{2}}}, \qquad 3.18$$

where $n = 0,1,2,3, \ldots$ .

This function is further examined in Appendix II.

3.7 Equation (3.14) can be further simplified:

$$\bar{E}(x) = \frac{\bar{G}a^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} \frac{\left(\frac{n+1}{2}\right)}{\left(\frac{n+1}{2}\right)} \Gamma\left(\frac{n+1}{2}\right) \frac{J_{\frac{n+1}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{n+1}{2}}}. \qquad 3.19$$

---

* $\Lambda_n(x)$ for $n = 1,2,3, \cdots ,10$ are tabulated and plotted in Appendix V.

** Bessel Functions, (7), pp 14.

And, using the definition given in (3.18), (3.13) can be written:

$$\bar{E}(x) = \frac{\bar{G}a^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} \frac{\Lambda_{\frac{n+1}{2}}(x)}{\frac{n+1}{2}} \quad . \qquad 3.19$$

The selection of exponents and the order of constants in the expansion was based on the knowledge that $\Lambda_{\frac{1}{2}}(x)$ is the first of the type that can occur. Integration of (3.13), which results in (3.19), requires that n be $> -1$, hence $n = 0$ is the first term which can be considered. And the case for $n = 0$ results in $\Lambda_{\frac{1}{2}}(x)$. Equation (3.19) is an important result, for the integration of (3.16) has given a single summation of $\Lambda$ functions of the type:

$$\bar{E}(x) = \frac{\bar{G}a^2}{2}\left[I_{\frac{1}{2}} \frac{\Lambda_{\frac{1}{2}}(x)}{\frac{1}{2}} + I_1 \frac{\Lambda_1(x)}{1} + I_{\frac{3}{2}} \frac{\Lambda_{\frac{3}{2}}(x)}{\frac{3}{2}} + \cdots\right]. \qquad 3.20a$$

If the function $V_{\frac{n+1}{2}}(x)$ is defined*:

$$V_{\frac{n+1}{2}}(x) = \frac{\Lambda_{\frac{n+1}{2}}(x)}{\frac{n+1}{2}} \quad , \qquad 3.21$$

then (3.20) becomes:

$$\bar{E}(x) = \frac{\bar{G}a^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} V_{\frac{n+1}{2}}(x). \qquad 3.22$$

Since a large number of distant field patterns occurring in nature are just such expressions, solutions for such patterns become very simple. Once the $I_{\frac{n+1}{2}}$ are all known, substitution into (3.07) gives the solution of the aperture distribution. The extreme simplicity of this method of solution is illustrated by several examples in Appendix III.

3.8 With the results of paragraph (3.7), the problem has become

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* $V_{\frac{n+1}{2}}(x)$ for $n = 0, 1, 2, 3, \cdots 9$ are plotted and tabulated in Appendix V.

one of expressing the given distant field pattern as a series of $V$ functions. Once this has been done, as paragraph (3.7) shows, the problem is virtually solved. In line with this problem, the $\Lambda$ and $V$ functions have been investigated by the author, and the results of the study compiled in Appendices II and V.

3.9 A summary of the method proposed by this chapter follows:

Given: Distant field pattern, $\bar{E}(x)$.

a. Express $\bar{E}(x)$ explicitly in V functions:

$$\bar{E}(x) = \frac{\bar{G}a^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} V_{\frac{n+1}{2}}(x). \qquad 3.21$$

b. Using $I_{\frac{n+1}{2}}$ in the expression for $\bar{E}(x)$, form the expression of the solution:

$$I(z) = \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} (1-z^2)^{\frac{n-1}{2}}. \qquad 3.07$$

c. $I(z)$ is then the required aperture distribution to set up the given $\bar{E}(x)$.

d. $I(z)$ may be conveniently plotted by using Table 3, Appendix V, Powers of $(1-z^2)^n$.

$$E(x) = \frac{2}{\sqrt{2q_\nu}} \sum_{n=0}^{\infty} I_{n+\frac{1}{2}} \Lambda_{n+\frac{1}{2}}(x).$$

$$I(\xi) = \sum_{n=0}^{\infty} I_{n+\frac{1}{2}} (1-\xi_1)^{\frac{1}{2}}.$$

## Chapter IV

### A Method for Solution of Aperture Distribution by a Mathematical Device

4.1 The difficulty of expressing the distant field pattern $E(x)$ as a series in V functions (necessary for the method of Chapter III) has led the author to a method of solution by mathematical devices. This method is based on Poisson's Integral*,

$$\Gamma(\tfrac{1}{2})\Gamma(n+\tfrac{1}{2})\frac{J_n(z)}{\left(\frac{z}{2}\right)^n} = \int_{-1}^{+1} e^{izt}(1-t^2)^{n-\frac{1}{2}}dt\,, \qquad 4.01$$

the generalized Lipschitz-Hankel Integral **,

$$\int_0^\infty e^{-at}J_\nu(bt)\,t^{\mu-1}dt = \frac{\left(\frac{b}{2a}\right)^\nu \Gamma(\mu+\nu)}{a^\mu\,\Gamma(\nu+1)}\,{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \nu+1; -\frac{b^2}{a^2}\right), \qquad 4.02$$

$$[\mathcal{R}(\mu+\nu) > 0]\,,$$

and the derivation of the distant field pattern as presented in Chapter III***.

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* Watson (5): Equation 7, page 25.

** Watson (5): Page 384. ${}_2F_1$ is a notation for the hypergeometric series: ${}_2F_1(\alpha,\beta;\rho;z) = \sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{n!\,(\rho)_n}z^n$,
where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$, $(\alpha)_0 = 1$.
This series is discussed in Watson (5), page 190.

*** For a circularly symmetric aperture distribution, it has been proved in Chapter III, equations (3.12) and (3.14), that a term in the expression of the aperture distribution, $I_n(z) = I_n(1-z^2)^{n-1}$ causes the term $E_n(x) = \frac{Ga^2 I_n \Gamma(n)}{2}\frac{J_n(x)}{\left(\frac{x}{2}\right)^n}$ to exist in the expression of the distant field pattern.

$$\mathcal{L}\{t^{\lambda}J_{\mu}(at);s\} = \int_0^{\infty} e^{-st} t^{\lambda} J_{\mu}(at)\, dt = \frac{a^{\mu}\,\Gamma(\lambda+\mu+1)}{2^{\mu}\, s^{\lambda+\mu+1}\,\Gamma(\mu+1)}\, {}_2F_1\!\left(\frac{\lambda+\mu+1}{2}, \frac{\lambda+\mu+2}{2}; \mu+1; -\frac{a^2}{s^2}\right),$$

$$\left[\Re(\lambda+\mu) > -1\right].$$

$$\mathcal{L}\{t^{\mu-\frac{1}{2}}J_{\mu}(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-\frac{1}{2}} J_{\mu}(at)\, dt = \frac{(2a)^{\mu}}{a\sqrt{\pi}}\, \Gamma\!\left(\mu+\tfrac{1}{2}\right) (s^2+a^2)^{-\mu-\frac{1}{2}},$$

$$\left[\Re(\mu) > -\tfrac{1}{2}\right].$$

$$\mathcal{L}\{t^{\mu-1}J_{\mu}(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} J_{\mu}(at)\, dt = \frac{a^{\mu}}{2^{\mu}\, s^{\mu}}\, {}_1F_0\!\left(\mu; -\frac{a^2}{s^2}\right),$$

$$\left[\Re(s) > |\Im(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}I_{\mu}(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} I_{\mu}(at)\, dt = \frac{a^{\mu}}{2^{\mu}\, s^{\mu}}\, {}_1F_0\!\left(\mu; \frac{a^2}{s^2}\right),$$

$$\left[\Re(s) > |\Re(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}e^{-at};s\} = \int_0^{\infty} e^{-st} t^{\mu-1} e^{-at}\, dt = \frac{1}{(s+a)^{\mu}},$$

$$\left[\Re(s) > -\Re(a)\right].$$

$$\mathcal{L}\{t^{\mu-1}\cosh(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \cosh(at)\, dt = \frac{1}{2}\left\{\frac{1}{(s-a)^{\mu}} + \frac{1}{(s+a)^{\mu}}\right\},$$

$$\left[\Re(s) > |\Re(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}\sinh(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \sinh(at)\, dt = \frac{1}{2}\left\{\frac{1}{(s-a)^{\mu}} - \frac{1}{(s+a)^{\mu}}\right\},$$

$$\left[\Re(s) > |\Re(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}\cos(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \cos(at)\, dt = \frac{1}{2}\left\{\frac{1}{(s-ia)^{\mu}} + \frac{1}{(s+ia)^{\mu}}\right\},$$

$$\left[\Re(s) > |\Im(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}\sin(at);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \sin(at)\, dt = \frac{1}{2i}\left\{\frac{1}{(s-ia)^{\mu}} - \frac{1}{(s+ia)^{\mu}}\right\},$$

$$\left[\Re(s) > |\Im(a)|\right].$$

$$\mathcal{L}\{t^{\mu-1}\log t;s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \log t\, dt = \frac{\Gamma(\mu)}{s^{\mu}}\left\{\psi(\mu) - \log s\right\},$$

$$\left[\Re(s) > 0,\ \Re(\mu) > 0\right].$$

$$\mathcal{L}\{t^{\mu-1}\log(t+a);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \log(t+a)\, dt = \frac{\Gamma(\mu)}{s^{\mu}}\left\{\log a + \psi(\mu) - \frac{a}{\mu}\, {}_2F_1(1,1;\mu+1;-a)\right\},$$

$$\left[\Re(s) > 0,\ \Re(\mu) > 0,\ |\arg a| < \pi\right].$$

$$\mathcal{L}\{t^{\mu-1}\log(t^2+a^2);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \log(t^2+a^2)\, dt = \frac{\Gamma(\mu)}{s^{\mu}}\left\{2\log a + 2\psi(\mu) - \frac{a^2}{\mu}\, {}_2F_1(1,1;\mu+1;-a^2)\right\},$$

$$\left[\Re(s) > 0,\ \Re(\mu) > 0,\ |\arg a| < \pi\right].$$

$$\mathcal{L}\{t^{\mu-1}\log(t^2-a^2);s\} = \int_0^{\infty} e^{-st} t^{\mu-1} \log(t^2-a^2)\, dt = \frac{\Gamma(\mu)}{s^{\mu}}\left\{2\log a + 2\psi(\mu) - \frac{a^2}{\mu}\, {}_2F_1(1,1;\mu+1;a^2

4.3 Poisson's integral (4.01) may be written:

$$\frac{Ga^2 I_n \Gamma(n) J_n(x)}{\left(\frac{x}{2}\right)^n} = \frac{Ga^2 \Gamma(n) I_n}{2\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{+1} e^{ixz} (1-z^2)^{n-\frac{1}{2}} dz, \qquad 4.13$$

or by Chapter III:

$$E_n(x) = \frac{Ga^2 \Gamma(n)}{2\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{+1} e^{ixz} I_n(z) (1-z^2)^{\frac{1}{2}} dz. \qquad 4.14$$

4.5 The hypergeometric series of the Lipschitz-Hankel integral (4.02) is difficult to handle, but it was observed that a specialized form of the integral† in the case $\mu=1$ and $\nu$ is any integer becomes:

$$\int_0^\infty e^{-at} J_\nu(bt) \frac{dt}{t} = \frac{\{\sqrt{a^2+b^2} - a\}^\nu}{\nu b^\nu}. \qquad 4.15$$

Let $a=iz$, $b=1$, $t=x$, and $\nu=n$. Then (4.15) may be written

$$\int_0^\infty e^{-ixz} \frac{J_n(x)}{x} dx = \frac{\{\sqrt{1-z^2} - iz\}^n}{n}. \qquad 4.16$$

4.4 Now we may let $n=1$ in (4.06), and we interest ourselves only in the real part of the value of the integrand

$$\mathcal{R} \int_0^\infty e^{-ixz} \frac{J_1(x)}{x} dx = (1-z^2)^{\frac{1}{2}}, \qquad 4.17$$

which may be written

$$\mathcal{R} \int_0^\infty e^{-ixz} \left[\frac{G}{2} a^2 I_1 \Gamma(1) \frac{J_1(x)}{\left(\frac{x}{2}\right)}\right] dx = Ga^2 \Gamma(1) I_1 (1-z^2)^{\frac{1}{2}}. \qquad 4.18$$

And by Chapter III, (4.18) becomes:

$$\mathcal{R} \int_0^\infty e^{-ixz} E_1(x) dx = Ga^2 \Gamma(1) I_1(z) (1-z^2)^{\frac{1}{2}}. \qquad 4.19$$

---

† Watson (5), Formula 7, page 386.

4.3 To evaluate $\mathcal{R}\int_0^\infty e^{-ixz}\,\frac{J_2(x)}{x^2}dx$ we may use the recurrence formula*

$$\frac{J_n(x)}{x} = \frac{1}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]. \qquad 4.10$$

In (4.10) we let $n = 2$ and multiply both sides of the equation by $x^{-1}$, obtaining

$$\frac{J_2(x)}{x^2} = \frac{1}{4}\left[\frac{J_1(x)}{x} + \frac{J_3(x)}{x}\right]. \qquad 4.11$$

Now

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx = \frac{\mathcal{R}}{4}\left[\int_0^\infty e^{-ixz}\frac{J_1(x)}{x}dx + \int_0^\infty e^{-ixz}\frac{J_3(x)}{x}dx\right], \qquad 4.12$$

and the right hand side of (4.12) is easily integrated by application of (4.06).

$$\therefore \mathcal{R}\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx = \frac{1}{4}\left[(1-z^2)^{\frac{1}{2}}\right] + \frac{\mathcal{R}}{4}\frac{\left[(1-z^2)^{\frac{1}{2}} - iz\right]^3}{3}$$

$$= \frac{1}{3}\,(1-z^2)^{\frac{3}{2}}. \qquad 4.13$$

By the methods of Chapter III, (4.13) may now be written:

$$\mathcal{R}\int_0^\infty e^{-ixz}E_2(x)\,dx = \frac{2}{3}\,6a^2\,\Gamma(2)\,I_2(z)\,(1-z^2)^{\frac{1}{2}}. \qquad 4.14$$

4.4 $\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_3(x)}{x^3}dx$ will now be evaluated. The recurrence formula (4.10), $n = 3$, becomes

$$\frac{J_3(x)}{x} = \frac{1}{6}\left[J_2(x) + J_4(x)\right]. \qquad 4.15$$

By multiplying both sides of equation (4.15) by $x^{-2}$, an expression is obtained for $\frac{J_3(x)}{x^3}$. Therefore

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_3(x)}{x^3}dx = \frac{\mathcal{R}}{6}\left[\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx + \int_0^\infty e^{-ixz}\frac{J_4(x)}{x^2}dx\right]. \qquad 4.16$$

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* Watson (5). Page 17.

$$\mathcal{R}\int_0^\infty e^{-ix\xi}\frac{x_3}{J^3(x)}dx = \frac{e}{6}\mathcal{R}\left[\int_0^\infty e^{-ix\xi}\frac{x_2}{J'(x)}dx + \int_0^\infty e^{-ix\xi}\frac{x_2}{J^4(x)}dx\right].$$

$$\frac{x}{J^5(x)} = \frac{e}{1}\left[J^3(x) + J^4(x)\right].$$

$$= \mathcal{R}\int_0^\infty e^{-ix\xi}\frac{x_3}{J^3(x)}dx$$

$$\mathcal{R}\int_0^\infty e^{-ix\xi}E^7(x)dx = \frac{3}{2}\,e\sigma\sigma_7 L(z)I^3(\xi)\,(1-\xi_1^2)^{\frac{7}{2}}.$$

$$= \frac{3}{1}\,(1-\xi_1)^{\frac{3}{2}}.$$

$$\therefore \mathcal{R}\int_0^\infty e^{-ix\xi}\frac{x_2}{J^2(x)}dx = \frac{4}{1}\left[(1-\xi_1)^{\frac{1}{2}}\right] + \frac{4}{\mathcal{R}}\left[\overline{(1-\xi_1)^{\frac{3}{2}}} - i\overline{\xi}\right]_2$$

$$\mathcal{R}\int_0^\infty e^{-ix\xi}\frac{x_2}{J^2(x)}dx = \frac{4}{\mathcal{R}}\left[\int_0^\infty e^{-ix\xi}\frac{x}{J'(x)}dx + \int_0^\infty e^{-ix\xi}\frac{x}{J^3(x)}\,dx\right],$$

$$\frac{x_2}{J^2(x)} = \frac{4}{1}\left[\frac{x}{J'(x)} + \frac{x}{J^3(x)}\right].$$

$$= \frac{x}{J^u(x)} = \frac{2u}{1}\left[J^{u-1}(x) + J^{u+1}(x)\right].$$

$$\mathcal{R}\int_0^\infty e^{-ix\xi}\frac{x_r}{J^r(x)}dx$$

Again using the recurrence formula (4.10) and the method of paragraph 5

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_4(x)}{x^2}dx = \frac{\mathcal{R}}{8}\left[\int_0^\infty e^{-ixz}\frac{J_3(x)}{x}dx + \int_0^\infty e^{-ixz}\frac{J_5(x)}{x}dx\right]. \qquad 4.17$$

Substituting, (4.16) becomes

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_3(x)}{x^3}dx = \frac{\mathcal{R}}{6}\left[\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx + \frac{1}{8}\left\{\int_0^\infty e^{-ixz}\frac{J_1(x)}{x}dx + \int_0^\infty e^{-ixz}\frac{J_5(x)}{x}dx\right\}\right]. \qquad 4.18$$

When we evaluate the first integral on the right side by (4.13) and the second and third by (4.06), we observe:

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_3(x)}{x^3}dx = \frac{1}{6}\left[\frac{1}{3}(1-z^2)^{\frac{3}{2}} + \frac{1}{8}\left\{\frac{\mathcal{R}\left[(1-z^2)^{\frac{1}{2}}-iz\right]^3}{3} + \frac{\mathcal{R}\left[(1-z^2)^{\frac{1}{2}}-iz\right]^5}{5}\right\}\right]$$

$$= \frac{1}{15}(1-z^2)^{\frac{1}{2}}(1-z^2)^2. \qquad 4.19$$

By the method of Chapter III, (4.19) may be expressed

$$\mathcal{R}\int_0^\infty e^{-ixz}E_3(x)dx = \frac{4}{15}Ga^2\Gamma(3)I_3(x)(1-z^2)^{\frac{1}{2}}. \qquad 4.20$$

In similar fashion, it may be shown that

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_4(x)}{x^4}dx = \frac{1}{105}(1-z^2)^{\frac{1}{2}}(1-z^2)^3. \qquad 4.21$$

4.7 Utilizing the results expressed in (4.07), (4.13), (4.19) and (4.21), the following general formula is derived by induction:*

$$\mathcal{R}\int_0^\infty e^{-ixz}\frac{J_n(x)}{x^n}dx = \frac{(1-z^2)^{\frac{1}{2}}(1-z^2)^{n-1}}{[1\cdot3\cdot5\cdots(2n-1)]}, \qquad 4.22$$

which may be written:

$$\mathcal{R}\int_0^\infty e^{-ixz}E_n(x)dx = \frac{2^{n-1}Ga^2\Gamma(n)I_n(z)(1-z^2)^{\frac{1}{2}}}{[1\cdot3\cdot5\cdots(2n-1)]}. \qquad 4.23$$

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* n is considered only as an integer in this chapter development.

$$\mathcal{L}\int_0^\infty e^{-ix\xi} E^\nu(x)\,dx = \frac{[1\cdot 3\cdot 5\cdots(2\nu-1)]}{\sum_{\nu-1}(\alpha^{-\nu}_{\xi}\,L(\nu)\,I^\nu(\epsilon)\,(1-\epsilon_1)^{\frac{1}{2}}}\,.$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi}\frac{x_\nu}{I^\nu(x)}\,dx = \frac{[1\cdot 3\cdot 5\cdots(2\nu-1)]}{(1-\epsilon_1)^{\frac{1}{2}}\,(1-\epsilon)^{\nu-1}}\,.$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi}\frac{x_4}{I^4(x)}\,dx = \frac{16\varepsilon}{1}\,(1-\epsilon_1)^{\frac{1}{2}}\,(1-\epsilon_1)^{\frac{3}{2}}\,.$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi} E^3(x)\,dx = \frac{15}{4}\,(-\alpha_\xi L(3)\,I^3(x)\,(1-\epsilon_1)^{\frac{1}{2}}\,.$$

$$= \frac{15}{1}\,(1-\epsilon_1)^{\frac{1}{2}}\,(1-\epsilon_1)^{\frac{5}{2}}\,.$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi}\frac{x_3}{I^3(x)}\,dx = \frac{e}{1}\left[\frac{3}{1}\,T(1-\epsilon_1)^{\frac{1}{2}} + \frac{8}{1}\left\{\mathcal{R}\left[(1-\epsilon_1)^{\frac{3}{2}}-i\epsilon\right] + \mathcal{R}\left[(1-\epsilon_1)^{\frac{5}{2}}-i\epsilon\right]\right\}\right]$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi}\frac{x_2}{I^2(x)}\,dx = \frac{e}{I^2}\left[\int_0^\infty e^{-ix\xi}\frac{x_2}{I^3(x)}\,dx + \frac{8}{1}\left\{\int_0^\infty e^{-ix\xi}\frac{x}{I^2(x)}dx + \int_0^\infty e^{-ix\xi}\frac{x}{I^3(x)}dx\right\}\right].$$

$$\mathcal{L}\int_0^\infty e^{-ix\xi}\frac{x_1}{I^1(x)}\,dx = \frac{e}{I^2}\left[\int_0^\infty e^{-ix\xi}\frac{x}{I^2(x)}\,dx + \int_0^\infty e^{-ix\xi}\frac{x}{I^2(x)}\,dx\right].$$

4.6 Let $$\mathbb{I}_n(z) = \frac{Ga^2 2^{n-1}\Gamma(n) I_n (1-z^2)^{n-\frac{1}{2}}}{[1\cdot 3\cdot 5\cdots(2n-1)]}, \qquad 4.24$$

where $$\mathbb{I}_n = \frac{Ga^2 2^{n-1}\Gamma(n) I_n}{[1\cdot 3\cdot 5\cdots(2n-1)]}. \qquad 4.25$$

Then (4.22) becomes:

$$\mathcal{R}\int_0^{\infty} e^{-ixz} E_n(x)dx = \mathbb{I}_n(z), \qquad 4.26$$

and (4.04) may be written:

$$E_n(x) = \frac{[1\cdot 3\cdot 5\cdots(2n-1)]}{2^n \Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})} \int_{-1}^{+1} e^{ixz}\mathbb{I}_n(z)\,dz. \qquad 4.27$$

In a sense (4.26) and (4.27) are a transform pair, but this is not to be their application in the proposed method. (4.26) is to be considered merely as a mathematical device, and (4.27) will not be of further use.

4.3 Now $E(x) = \sum_{n=1}^{\infty} E_n(x)$, and therefore (4.26) may be extended:

$$\mathcal{R}\int_0^{\infty} e^{-ixz} E(x)dx = \sum_{n=1}^{\infty} \mathbb{I}_n(z). \qquad 4.28$$

Since it was proved in Chapter III that a circularly symmetric aperture distribution results in a distant field pattern expressible as a summation of $E_n(x) = \frac{Ga^2}{2}\Gamma(n) I_n \frac{J_n(x)}{(\frac{x}{2})^n}$ for $n = 1,2,3,\cdots$, then any distant field pattern which is capable of being produced by a circularly symmetric aperture distribution must satisfy (4.28) regardless of how this distant field pattern is expressed. This is very important due to our present inability to express an arbitrary $E(x)$ as a summation of V functions. It must, however, be repeated here that (4.28) is a mathematical device and not a physical representation of the radiation problem. The independent variable "x" in the expression for $E(x)$ as derived in Chapter III is

as $\sin\theta$ and thus is completely defined from 0 to $\frac{2\pi a}{\lambda}$. The contribution for any x greater than this value $\frac{2\pi a}{\lambda}$ must be considered zero (actually such values of x are meaningless). But in (4.16), x has values from 0 to $\infty$. Therefore, for correct use of (4.16), h(x) must express the distant field pattern for values of x from 0 to $\frac{2\pi a}{\lambda}$ and must, in addition, maintain its V function expression for values of x from $\frac{2\pi a}{\lambda}$ to $\infty$.

4.10 Thus far we have considered only the real part of $\int_0^\infty e^{-ixz} E_n(x)\,dx$ as in equation (4.16). To formulate a method for working practical problems, it becomes necessary to examine the complete complex value of the integral. We shall first examine the Lipschitz-Hankel integral (4.16) with n=1.

$$\int_0^\infty e^{-ixz}\,\frac{J_1(x)}{x}\,dx \;=\; \left[(1-z^2)^{\frac{1}{2}} - iz\right] \qquad 4.29$$

Expressing $\frac{J_1(x)}{x}$ in asymptotic series*, the left hand side of (4.29) becomes:

$$\int_0^\infty e^{-ixz}\left[\frac{1}{2} - \frac{x^2}{16} + \frac{x^4}{24\cdot 16} - \frac{x^6}{12\cdot 24\cdot 64} + \cdots\right] dx\,.$$

Utilizing the following integration formula

$$\int_0^\infty e^{-ixz}\,x^{2p}\,dx \;=\; \frac{(2p)!}{(iz)^{2p+1}}\,, \qquad 4.30$$

the integral can be evaluated:

$$\int_0^\infty e^{-ixz}\,\frac{J_1(x)}{x}\,dx = \left[\frac{1}{2(iz)} - \frac{1}{8(iz)^3} + \frac{1}{16(iz)^5} - \frac{5}{128(iz)^7} + \cdots\right]. \qquad 4.31$$

As expected from (4.29), the imaginary part of the integration performed has cancelled the first term, (iz), in the binomial theorem

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* From (4.17): $\frac{J_1(x)}{x} = \frac{1}{2} - \frac{x^2}{16} + \frac{x^4}{24\cdot 16} - \frac{x^6}{12\cdot 24\cdot 64} + \cdots$ .

$$\frac{x}{J(x)} = \frac{2}{1} - \frac{16}{x^2} + \frac{24\cdot16}{x^4} - \frac{12\cdot24\cdot64}{x^6} + \cdots ,$$

$$\int_0^\infty e^{-ix\varepsilon}\,\frac{x}{J(x)}\,dx = \left[\frac{2(i\varepsilon)}{1} - \frac{8(i\varepsilon)^2}{1} + \frac{16(i\varepsilon)^3}{1} - \frac{128(i\varepsilon)^4}{5} + \cdots\right] .$$

$$\int_0^\infty e^{-ix\varepsilon}\,x^{2p}\,dx = \frac{(i\varepsilon)^{2p+1}}{(2p)!} ,$$

$$\int_0^\infty e^{-ix\varepsilon}\left[\frac{2}{1} - \frac{16}{x^2} + \frac{24\cdot16}{x^4} - \frac{12\cdot24\cdot64}{x^6} + \cdots\right] dx .$$

$$\frac{x}{J(x)}$$

$$\int_0^\infty e^{-ix\varepsilon}\,\frac{x}{J(x)}\,dx = \left[(1-\varepsilon^2)^{\frac{1}{2}} - i\varepsilon\right]$$

$$\int_0^\infty e^{-ix\varepsilon}\,E^{\nu}(x)\,dx$$

$$\frac{\sqrt{\pi}}{(2\varepsilon)^{\nu}} \qquad \infty$$

$$\infty$$

$$\frac{\sqrt{\pi}}{2\Gamma a} \qquad \varepsilon =$$

expansion of $(1-z^2)^{\frac{1}{2}}$*. Now let us examine the complete integrals of (4.12).

$$\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx = \frac{1}{4}\left[\{(1-z^2)^{\frac{1}{2}}-iz\} + \frac{\{(1-z^2)^{\frac{1}{2}}-iz\}^3}{3}\right] \qquad 4.18$$

By the use of (3.17) $\frac{J_2(x)}{x^2}$ can be expressed in asymptotic series, and the left side of (4.18) may be written

$$\int_0^\infty e^{-ixz}\left[\frac{1}{8}-\frac{x^2}{8\cdot12}+\frac{x^4}{8\cdot24\cdot16}-\frac{x^6}{24\cdot120\cdot64}+\cdots\right]dx.$$

Integration by the use of (4.10) gives

$$\int_0^\infty e^{-ixz}\frac{J_2(x)}{x^2}dx = \left[\frac{1}{8(iz)}-\frac{1}{48(iz)^3}+\frac{1}{8\cdot16(iz)^5}-\frac{1}{4\cdot64(iz)^7}+\cdots\right]. \qquad 4.19$$

It is apparent that the imaginary component of the integration has now cancelled the first two terms in the expansion of $(1-z^2)^{\frac{3}{2}}$**. In examination of $\int_0^\infty e^{-ixz}\frac{J_3(x)}{x^3}dx$ it is observed that the imaginary contribution has cancelled the first three terms in the expansion of $(1-z^2)^{\frac{5}{2}}$***. In general, the imaginary part in the integration of $\int_0^\infty e^{-ixz}\frac{J_n(x)}{x^n}dx$ cancels the first n terms of the binomial expansion of $(1-z^2)^{n-\frac{1}{2}}$.

4.11 Paragraph (4.10) has pointed out the practical method sought in this chapter. For if the ratio in the binomial theorem expansion of $(1-z^2)^{n-\frac{1}{2}}$ of the $(n+1+k)$th terms to each other is the

---

* $(1-z^2)^{\frac{1}{2}} = ([-z^2]+1)^{\frac{1}{2}}$

$= [-z^2]^{\frac{1}{2}}+\left(\frac{1}{2}\right)[-z^2]^{-\frac{1}{2}}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{[-z^2]^{-\frac{3}{2}}}{2!}+\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{[-z^2]^{-\frac{5}{2}}}{3!}+\cdots$

$= \left\{[iz]\right\}+\frac{1}{2[iz]}-\frac{1}{8[iz]^3}+\frac{1}{16[iz]^5}-\frac{5}{128[iz]^7}+\cdots.$

** $(1-z^2)^{\frac{3}{2}} = \left\{[iz]^3+\frac{3}{2}[iz]\right\}+\frac{3}{8[iz]}-\frac{1}{16[iz]^3}+\frac{3}{16\cdot8[iz]^5}+\cdots.$

*** $(1-z^2)^{\frac{5}{2}} = \left\{[iz]^5+\frac{5}{2}[iz]^3+\frac{15}{8}[iz]\right\}+\frac{5}{16[iz]}-\frac{5}{16\cdot8[iz]^3}+\frac{3}{32\cdot8[iz]^5}+\cdots.$

Note: Brackets { } in footnotes above indicate portion of expansion cancelled in the integration by the imaginary part of the complex value.

$$\{\}$$

$$(1-\varepsilon_2)^{\frac{3}{2}} = \left\{[i\varepsilon]^2 + \tfrac{3}{8}[i\varepsilon]^3 + \tfrac{3}{128}[i\varepsilon]^4\right\} + \frac{16\,[i\varepsilon]}{2} - \frac{16\cdot 8\,[i\varepsilon]^3}{2} + \frac{32\cdot 8\,[i\varepsilon]^5}{3} + \cdots .$$

$$(1+\varepsilon_2)^{\frac{3}{2}} = \left\{[i\varepsilon]^2 + \tfrac{3}{8}[i\varepsilon]^3\right\} + \frac{8\,[i\varepsilon]}{3} - \frac{16\,[i\varepsilon]^3}{1} + \frac{16\cdot 8\,[i\varepsilon]^5}{3} + \cdots .$$

$$= \left\{[i\varepsilon]^2\right\} + \frac{2[i\varepsilon]^3}{1} - \frac{8\,[i\varepsilon]^3}{1} + \frac{16\,[i\varepsilon]^3}{1} - \frac{16\cdot 8\,[i\varepsilon]^5}{2} + \cdots .$$

$$= [-\varepsilon_2]^{\frac{2}{2}} + (\tfrac{3}{1})(-\varepsilon_2)^{\frac{1}{2}} + (\tfrac{3}{2})(-\tfrac{1}{2})\frac{(-\varepsilon_2)^{\frac{2}{2}}}{2!} + (\tfrac{3}{3})(-\tfrac{1}{2})(-\tfrac{3}{2})\frac{(-\varepsilon_2)^{\frac{3}{2}}}{3!} + \cdots$$

$$(1-\varepsilon_2)^{\frac{3}{2}} = \left([-\varepsilon_2]+1\right)^{\frac{3}{2}}$$

$$\varepsilon_2^{\,\frac{3}{2}}$$

$$\int_0^\infty e^{-ix\varepsilon}\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}\,dx$$

$$\varepsilon_2^{\,\frac{3}{2}}$$

$$\int_0^\infty e^{-ix\varepsilon}\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}\,dx = \left[\frac{1}{\Gamma(i\varepsilon)} - \frac{48\,(i\varepsilon)}{1} + \frac{8\cdot 16\,(i\varepsilon)^2}{1} - \frac{4\cdot 64\,(i\varepsilon)^3}{1} + \cdots\right].$$

$$\int_0^\infty e^{-ix\varepsilon}\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}\,dx = \left[\frac{1}{8} - \frac{8\cdot 16}{x_2} + \frac{8\cdot 24\cdot 16}{x_4} - \frac{24\cdot 120\cdot e_4}{x_6} + \cdots\right]dx .$$

$$\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}$$

$$\int_0^\infty e^{-ix\varepsilon}\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}\,dx = \frac{4}{\Gamma}\left[\{(1-\varepsilon_2)^{\frac{1}{2}} - i\varepsilon\} + \frac{\{(1-\varepsilon_2)^{\frac{3}{2}} - i\varepsilon\}^2}{2}\right]$$

$$\frac{x_2^{\frac{3}{2}}}{\Gamma^2(x)}$$

ratio of the $k$th terms in the integration of

$$\int_0^\infty e^{-ixz}\left[\sum_{k=0}^{\infty} F_k (x)^{2k}\right]dx = \sum_{k=0}^{\infty} F_k \frac{(2k)!}{(iz)^{2k+1}} \qquad 4.34$$

$[k = 0,1,2,\cdots ; n = 1,2,3,\cdots]$

to each other, where

$$\sum_{k=0}^{\infty} F_k (x)^{2k} = E(x) , \qquad 4.35$$

then it is known that $(1-z^2)^{n-\frac{1}{2}}$ is the only factor of $\mathcal{R}\int_0^\infty e^{-ixz}\left[\sum_{k=0}^{\infty} F_k(x)^{2k}\right]dx$. Hence, it is known from (4.16) and (4.34) that $I_n(1-z^2)^{n-1}$ is the aperture distribution. If on the other hand, none of the $(1-z^2)^{n-\frac{1}{2}}$ tried fit exactly, this means that $E(x)$ contains more than one $V_n(x)$ in its expression. Then simultaneous equations must be used, considering enough items to either actually obtain a solution or approximate a solution to a sufficient degree of accuracy. When the aperture distribution is assumed $I(z) = \sum_{n=1}^{p} I_n (1-z^2)^{n-1}$, the simultaneous equations that must be considered for a complete solution are*:

$$\sum_{k=0}^{p-1}\left[F_k(2k)! - \sum_{n=1}^{p} \mathrm{II}_n\left\{\frac{(n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2}-k)}{(n+k)!}\right\} = 0\right] . \qquad 4.36$$

---

* Equation (4.36) has been obtained by equating the sum of coefficients of $\frac{1}{(iz)^{2k+1}}$ in the binomial theorem expansions of the component terms $\mathrm{II}_n(1-z^2)^{n-\frac{1}{2}}$ in the aperture distribution to the coefficient of $\frac{1}{(iz)^{2k+1}}$ in the integration of (4.14), $k = 0,1,2,3,\cdots$. If $p$ terms appear in the aperture distribution, of course $p$ equations must be used, and hence $k$ must be considered from 0 to $p-1$.

$$\int_0^\infty e^{-ix\xi}\left[\sum_{k=0}^{\infty} E^k(x)_{sk}\right]dx = \sum_{k=0}^{\infty} E^k \frac{(i\xi)^{sk+1}}{(sk)!}$$

$$[k = 0,1,2,\ldots;\ \nu = 1,2,3,\ldots]$$

$$\sum_{k=0}^{\infty} E^k(x)_{sk} = E(x)\ ,$$

$$\mathcal{F}\int_0^\infty e^{-ix\xi}\left[\sum_{k=0}^{\infty} E^k(x)_{sk}\right]dx \cdot \qquad \Sigma s_{\nu-\frac{1}{2}}$$

$$\sum_{k=0}^{s_\nu - 1}$$

$$I(\xi) = \sum_{b=1}^{\nu} I^\nu (1-\xi_1)^{\nu-1}$$

$$\sum_{b=1}^{\infty}{}' \left[E^k(sk)! - \sum_{\nu=1}^{b} \Pi^\nu \frac{(\nu+k)!}{(\nu-\frac{1}{2})(\nu-\frac{3}{2})\cdots(\frac{1}{2}-k)}\right] = 0 \Bigg] \cdot$$

$$\frac{(i\xi)^{sk+1}}{1}$$

$$\frac{\Pi^\nu}{(i\xi)^{sk+1}} \quad \xi - \frac{1}{2}$$

by the use of the $p$ equations of (4.28), the $p$ unknowns can be found. A trial and error method of selecting the "n" to use in (4.30) is the only method known at the present time. Observation of the character of a plotted $E(x)$ may lead one to make a good guess.

4.12 The method presented in this chapter is summarized in steps below:

a. Express the arbitrary $E(x)$ as a function of $x^2$ (4.15).

b. Multiply $E(x)$ by $e^{-ixz}dx$ and integrate over the limits 0 to $\infty$ by use of formula (4.30).

c. Set up simultaneous equations (4.28) and solve for $\mathrm{II}_n$.

d. Since by (4.26), $\mathrm{II}_n = \dfrac{Ga^2 2^{n-1}\Gamma(n) I_n}{[1.3.5\cdots(2n-1)]}$ solve for $I_n$.

e. The resultant aperture distribution is then

$$I(z) = \sum_{n=1}^{\infty} I_n (1-z^2)^{n-1}.$$

f. $I(z)$ may then be plotted by using Table 2, Appendix V.

Examples of the method of this chapter are worked in Appendix IV.

4.13 It is believed that the method of this chapter may be extended to apply to more general $E(x)$. While it has not been proved, it appears reasonable that a distant field pattern which is itself symmetrical about the extended reflector axis has resulted from a circularly symmetric aperture distribution. If this is the case, then any symmetrical distant field pattern would be capable of expression as in equation (3.24), and, therefore, the method of this chapter could be used to solve for the aperture distribution. Paragraph (4.8) requires that $E(x)$ as used must express the distant field pattern for values of $x$ from 0 to $\frac{2\pi a}{\lambda}$ and must also maintain its V function expression for values of $x$ from $\frac{2\pi a}{\lambda}$ to $\infty$. Defining $E(x) = 0$ for values of $x > \frac{2\pi a}{\lambda}$ should introduce little error into the results because of the very rapid convergence of the V functions to 0 with

$$\frac{\sqrt{2\pi}}{2\pi^{x}} \qquad \frac{\sqrt{2\pi}}{2\pi^{x}} \infty \quad \text{---} \quad =$$

$$\Pi^{x}$$

$$\infty \qquad \text{-1x5-} \qquad x$$

$$\Pi^{x} = \frac{\Gamma(1)\Gamma(2)\cdots(2x-1)}{(\ln s_{x}\zeta_{x-1}\Gamma(x)\,\Gamma^{x}} \cdot$$

$$\Gamma(s) = \sum_{x=1}^{\infty} \Gamma^{x}\ (1-s_{x})_{x-1} \cdot \qquad (5)$$

the increase of x. Hence, if E(x) can be expressed as a function of $(x^2)$ which is the same as the distant field pattern, or a close approximation to it, for values of x from 0 to $\frac{2\pi a}{\lambda}$ and defined equal to 0 for values of $x > \frac{2\pi a}{\lambda}$, the use of this E(x) in (4.33) should be allowable as follows:

$$\mathcal{R}\int_0^{\frac{2\pi a}{\lambda}} e^{-ixz} E(x)\,dx + 0 = \sum_{n=1}^{\infty} \mathrm{II}_n(z) \qquad 4.35$$

The use of the method of this chapter would then proceed from (4.36) in the regular fashion. The accuracy of the result obtained can easily be checked once the $I_n$ have been determined by comparing $\frac{Ga^2}{2}\sum_{n=1}^{\infty} I_n V_n(x)$ with the known E(x).

# Chapter V

## Conclusions

5.1 The development in Chapter II of the integral equation relating the circularly symmetric aperture distribution and the distant field radiation pattern (2.54) and the solution of this equation in Chapter III moved to the following important conclusion. Since all circularly symmetric aperture distributions may be expressed as

$$I(z) = \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} \left(1-z^2\right)^{\frac{n-1}{2}}; \qquad 3.07$$

then all distant field patterns set up by these circularly symmetric aperture distributions must be capable of expression as

$$E(x) = \frac{Ga^2}{2} \sum_{n=0}^{\infty} I_{\frac{n+1}{2}} V_{\frac{n+1}{2}}(x). \qquad 3.21$$

Actually, the explicit expression of an arbitrary E(x) as indicated has not been possible in all cases. It is not even certain that such an expression is possible for an arbitrary E(x), since only E(x) which are set up by circularly symmetric aperture distributions are known to satisfy equation (3.21).

5.2 The method presented in Chapter III, which can be used only when E(x) is explicitly expressed as in (3.22), is very simple.

a. Express E(x) explicitly in V functions as in (3.22).

b. Using $I_{\frac{n+1}{2}}$ of (2.42), express I(z) as in (3.07).

5.3 Chapter IV presents a method which utilizes the conclusion of paragraph (5.1) as its foundation. Knowing paragraph (5.1) to be true, a mathematical device is used to solve for I(z). This method has the advantage that E(x) need not be expressed explicitly in V

$$I(\varepsilon) = \sum_{\nu=0}^{\infty} I_{\nu+1}^{\frac{1}{2}} \quad (1-\varepsilon_s)^{\frac{1}{\nu+1}} \quad ;$$

$$E(x) = \sum_{\nu=0}^{\infty} I_{\nu+1}^{\frac{1}{2}} \Lambda_{\nu+1}^{\frac{1}{2}} (x) \cdot$$

$\frac{1}{\nu+1}$

functions, as in the method of Chapter III, but may be expressed in $(x^2)$. This is a very important step, because many $E(x)$ which actually do satisfy (3.28) may not be easily expressed explicitly in V functions. An obstacle still to be overcome in the use of this method is that it is not possible at the present time to recognize an $E(x)$ which does not satisfy (3.28), meaning that it is not set up by a circularly symmetric aperture distribution. The method of Chapter IV follows:

a. Express the arbitrary $E(x)$ as a function of $x^2$.

b. Multiply $E(x)$ so expressed by $e^{-ixz}dx$ and integrate over the limits 0 to $\infty$ by the use of

$$\int_0^\infty e^{-ixz} x^{2p}\, dx = \frac{(2p)!}{(iz)^{2p+1}} . \qquad 4.30$$

c. Set up simultaneous equations and solve for $\mathbb{I}_n$:

$$\sum_{K=0}^{p-1}\left[F_k(2K)! - \sum_{n=1}^{p}\mathbb{I}_n\left\{\frac{(n-\frac{1}{2})(n-\frac{3}{2})\cdots(\frac{1}{2}-k)}{(n+k)!}\right\} = 0\right]. \qquad 4.31$$

d. Solve for $I_n$:

$$\mathbb{I}_n = \frac{Ga^2\, 2^{n-1}\,\Gamma(n)}{[1.3.5\cdots(2n-1)]}\, I_n . \qquad 4.33$$

e. Express the aperture distribution: $I(z) = \sum_{n=1}^{\infty} I_n (1-z^2)^{n-1}$.

3.4 It is believed that any distant field pattern which is symmetrical about the extended reflector axis may be assumed to satisfy (3.41); that is, it may be assumed that this pattern resulted from a circularly symmetric aperture distribution. If this is true, the method of Chapter IV may be applied to more general $E(x)$ as proposed in paragraph (4.13).

$$\int_0^{\infty} e^{-ix\xi} x^{2p} \, dx = \frac{(i\xi)^{-2p+1}}{(-2p)!} \cdot$$

$$\Pi^{\nu}:$$

$$\sum_{k=0}^{p-1} \left[ \Gamma^{k}(2k)! - \sum_{\mu=1}^{p} \Pi^{\mu} \left\{ \frac{(\mu+k)!}{(\mu-\frac{1}{2})(\mu-\frac{3}{2})\cdots(\frac{1}{2}-k)!} \right\} = 0 \right] \cdot$$

$$\Pi^{\nu} = \frac{[1\cdot 3\cdot 5\cdots(2\nu-1)]}{2^{\nu+1} \sum_{\mu=1}^{\nu} \Pi^{\mu} \, \Gamma(\mu)} \, \Gamma^{\nu} \cdot$$

$$\Gamma(s) = \sum_{\nu=1}^{\infty} \Gamma^{\nu} \, (1-s)_{\nu-1} \cdot$$

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## Vita

The author was born in Farragut, Iowa on May 6, 1921. He graduated from the Farragut Consolidated High School in 1938, from Bentworth Military Academy in 1940, and from the United States Naval Academy in 1943. He received the degree of Bachelor of Science from the Naval Academy. Since graduation, he has served on active duty in the United States Navy. From 1947 to 1948, he attended the United States Navy Postgraduate School; and from 1948 to 1950, he attended the Johns Hopkins University.

## Appendix I

### Table of Symbols (MKS Units)

| Symbol | Definition |
|---|---|
| $a$ | radius of aperture |
| $\bar{a}$ | unit vector in an arbitrary direction, Par. (2.6) |
| $b$ | constant, Par. (4.12) |
| $c$ | velocity of light |
| $C$ | contour bounding the aperture |
| $C_3$ | contour bounding $S_3$, Par. (2.12) |
| $da$ | differential element over surface S |
| $\bar{ds}$ | vector differential element of length about contour C |
| $dv$ | differential element of volume V |
| $\bar{E}$ | vector intensity of electric field |
| $\bar{E}(X',Y',Z')$ | vector intensity of electric field at fixed point of observation |
| $\bar{E}(R,\varphi,\theta)$ | expression of $\bar{E}(X',Y',Z')$ in spherical coordinates |
| $\bar{E}(R,\theta)$ | $\bar{E}(R,\varphi,\theta)$ for circularly symmetric aperture distribution |
| $\bar{E}(x)$ | electric field vector distribution in distant space, $\bar{E}(x) = \bar{E}(R,\theta)$ |
| $E_n(x)$ | term in the expression of $E(x)$, Par. (4.9) |
| $e^{i\omega t}$ | time factor |
| $f$ | frequency |
| $\bar{F}$ | directrix, time and range factor of $\bar{E}(x)$ |
| $F_K$ | coefficient of $x^{2k}$ in the expression of $E(x)$, Par. (4.11) |
| $\bar{G}$ | directrix, time and range factor of $\bar{H}(x)$ |
| $\bar{H}$ | vector intensity of the magnetic field |
| $\bar{H}(X',Y',Z')$ | vector intensity of the magnetic field at fixed point of observation |
| $\bar{H}(R,\varphi,\theta)$ | $\bar{H}(X',Y',Z')$ expressed in spherical coordinates |

| | |
|---|---|
| $\bar{H}(R,\theta)$ | $\bar{H}(R,\varphi,\theta)$ for circularly symmetric aperture distribution |
| $\bar{H}(x)$ | magnetic field vector distribution in distant space, $\bar{H}(x) = \bar{H}(R,\theta)$ |
| $\tilde{\bar{H}}$ | complex conjugate of $\bar{H}$ |
| $i$ | square root of minus one |
| $\bar{I}$ | unit vector in X direction |
| $\bar{I}(\hat{\rho},a)$ | electric field vector distribution over circularly symmetric aperture. For circularly symmetric aperture distributions, $I(\hat{\rho}/a)$ completely describes the distribution. |
| $I(z)$ | circularly symmetric amplitude distribution of electric field over the aperture, $z = \hat{\rho}/a$ |
| $I_n(z)$ | $I_n(z) = I_n(1 - z^2)^{n-1}$, Par. (4.11) |
| $I_n$ | coefficient of $(1 - z^2)^{n-1}$ in expression of aperture distribution |
| $\mathrm{II}_n(z)$ | fictitious aperture distribution of Chapter IV, Par. (4.8) |
| $\mathrm{II}_n$ | coefficient of $(1 - z^2)^{n-\frac{1}{2}}$ in expression of fictitious aperture distribution |
| $\bar{J}$ | unit vector in Y direction |
| $J_n(x)$ | Bessel function of the first kind of order n |
| $\bar{J}$ | electric current density vector |
| $\bar{J}^*$ | fictitious magnetic current density vector |
| $k$ | phase constant, $k = \omega\sqrt{\epsilon\mu}$ |
| $\bar{k}$ | unit vector in Z direction |
| $\bar{K}$ | electric current surface density vector |
| $\bar{K}^*$ | fictitious magnetic current surface density vector |
| $\bar{h}(\hat{\rho},a)$ | magnetic field vector distribution over circularly symmetric aperture. For circularly symmetric aperture distributions, $h(\hat{\rho}/a)$ completely describes the distribution. |
| $M(z)$ | amplitude distribution of magnetic field over aperture, $z = \hat{\rho}/a$ |
| $\bar{n}$ | unit vector normal to source in direction of radiation |

| | |
|---|---|
| $\bar{n}_1$ | unit vector in the surface $S_1$, normal to $\bar{\alpha}$ and $\bar{n}$ |
| $\bar{r}$ | position vector from point (X,Y,Z) in the aperture to the point of observation (X',Y',Z') |
| $\bar{r}_o$ | unit vector directed from the source on the aperture towards (X',Y',Z') |
| $\bar{R}$ | position vector from center of the aperture to the fixed point of observation (X',Y',Z') |
| $r_1$ | radius of sphere circumscribed about (X',Y',Z') to exclude the singularity of $\Phi$, Par. (2.8) |
| $\mathcal{R}$ | real part of a complex quantity |
| S | regular surface in space used in Green's Theorem |
| $\bar{S}$ | complex Poynting vector, the mean value of the rate at which energy traverses a surface whose normal is an electromagnetic field is in the direction $(\bar{E}\times\bar{H})$ |
| $S_1$ | area inside C in plane of aperture |
| $S_2$ | area outside C in plane of aperture |
| $S_3$ | area inside $C_3$ in plane of aperture, Par. (2.12) |
| $S_i$ | surface of sphere of radius $r_1$ circumscribed about (X',Y',Z'), Par. (2.8) |
| V | closed region of space as used in Green's Theorem |
| $V_n(x)$ | V function of order n, $V_n(x) = \frac{\Lambda_n(x)}{n}$ |
| $W_n(x)$ | W function of order n, Appendix II |
| x | $x = ka \sin\theta$ |
| (X,Y,Z) | rectangular coordinate system of which $\bar{i}, \bar{j}$, and $\bar{k}$ are unit vectors |
| (X',Y',Z') | fixed point of observation, that is the point at which Z is measured in distant space, $(X',Y',Z') = (R,\varphi,\theta)$ |
| z | $z = \hat{\rho}/a$ |
| $\beta$ | angle between $\bar{R}$ and $\bar{\hat{\rho}}$ |
| $\Gamma(n)$ | Gamma function |
| $\varepsilon$ | permittivity of space |
| $\eta$ | impedance of space, $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ |

| | |
|---|---|
| $\lambda$ | free space wavelength, $\lambda = c/f$ |
| $\Lambda_n(x)$ | Lambda function of order n |
| $\mu$ | permeability of space |
| $\Pi(p)$ | Gauss's pi function, $\Pi(p) = \Gamma(p + 1)$ |
| $\rho$ | electric charge density |
| $\rho(X, Y, Z)$ | electric charge density distribution |
| $\rho^*$ | fictitious magnetic charge density |
| $\bar{\hat{\rho}}$ | vector in aperture plane from center of aperture to element of source, for circularly symmetric aperture distributions only $\hat{\rho}$ is considered |
| $\sigma$ | line density of electric charge |
| $\sigma(X, Y, Z)$ | electric charge line density distribution |
| $\sigma^*$ | fictitious line density of magnetic charge |
| $\hat{\varphi}$ | angular coordinate of element of source in the aperture |
| $\Phi$ | $\Phi = \frac{e^{-ikr}}{r}$ ; Green's function $= \bar{a}\frac{e^{-ikr}}{r}$ |
| $\psi$ | angle between $\bar{R}$ and $\bar{I}$ |
| $\omega$ | angular velocity |
| $\bar{\nabla}$ | vector differential operator, $\bar{\nabla} = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$ |
| $p!$ | factorial p, $p! = 1 \cdot 2 \cdot 3 \cdots p$ |
| $\infty$ | infinity |

Appendix II

# $\Lambda$ and $V$ functions

II. 1 The $\Lambda$ function as commonly defined has been given in (2.15) and (2.16), and this function has been extended to half-integer orders in (2.16):

$$\Lambda_{\frac{n+1}{2}}(x) = \Pi\left(\frac{n+1}{2}\right) \frac{J_{\frac{n+1}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{n+1}{2}}}, \qquad (2.16)$$

where $n = 0, 1, 2, 3, \cdots$.

But it becomes apparent that the $\Lambda$ function itself does not tell the complete story in the present problem. The integration by Sonine's formula always results in terms of $\frac{\Lambda_n(x)}{n}$, and this function will be called the $V$ function in this paper.

$$V_{\frac{n+1}{2}}(x) = \frac{\Lambda_{\frac{n+1}{2}}(x)}{\left(\frac{n+1}{2}\right)} . \qquad (2.17)$$

This definition helps explain the reason $\Lambda_0$ does not occur, for while $\Lambda_0 = J_0$ and is finite, $\frac{\Lambda_0(x)}{0} = V_0(x)$ is not finite. Hence $V_0(x)$ would not occur. An interesting item is the regular way in which $V_{\frac{1}{2}}(x)$, $V_1(x)$, $V_{\frac{3}{2}}(x)$, etc. occur. Of course, all of the $V$ functions except $V_{\frac{1}{2}}(x)$ and $V_1(x)$ converge more rapidly than do the corresponding $\Lambda$ functions. These functions are plotted and tabulated in Appendix V.

II.2 $\Lambda_n(x)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (2n+1)\frac{dy}{dx} + xy = 0$$

as does $\frac{G_n(x)}{x^n}$ where

$$G_n(x) = \frac{\pi}{2 \sin n\pi}\left\{J_{-n}(x) - e^{-in\pi} J_n(x)\right\}.$$

$$\Lambda \qquad \Lambda$$

$$\Lambda$$

$$\Lambda \qquad \Lambda \qquad \Lambda$$

$$\Lambda^{\circ} \qquad \Lambda^{\circ}(x) = \Lambda^{\circ}(x)$$

$$\Lambda^{\circ} = \Lambda^{\circ}$$

$$\Lambda^{\circ}(x) = \Lambda^{\circ}(x)

When n is an integer

$$G_n(x) = J_n(x)\left\{-\log\frac{x}{2} - \gamma + \frac{i\pi}{2}\right\}$$

$$= -Y_n(x) + J_n(x)\left\{.11593 + \frac{i\pi}{2}\right\}.$$

A peculiar thing about this is that $\frac{G_n(x)}{x^n}$ and $\frac{J_n(x)}{x^n}$ are independent when n is not an integer, but they are not independent when n is an integer*.

II.3 Recurrence formulae for $\Lambda$ functions**:

a) $$\Lambda_\nu(x) = \Lambda_{\nu-1}(x) + \frac{x^2\Lambda_{\nu+1}(x)}{4\nu(\nu+1)},$$

b) $$\Lambda_{\nu-1}(x) = \left[1 - \frac{x^2}{4\nu(\nu+1)}\right]\Lambda_{\nu+1}(x) - \frac{x^2\Lambda_{\nu+2}(x)}{4(\nu+1)(\nu+2)},$$

c) $$\Lambda_\nu(x)\Lambda_{-\nu-1}(x) - \frac{x^2\Lambda_{\nu+1}(x)}{4\nu(\nu+1)}\Lambda_{-\nu}(x) = 1.$$

II.4 Differentiation formulae:

a) $$\frac{d}{dx}\left(\frac{J_n(x)}{x^n}\right) = -\frac{J_{n+1}(x)}{x^n},$$

$$\therefore \frac{d}{dx}\left(\frac{2^n\Pi(n)J_n(x)}{nx^n}\right) = -\frac{2^n\Pi(n)J_{n+1}(x)}{nx^n},$$

$$\left[n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots\right]$$

$$\therefore \frac{d}{dx}\left[V_n(x)\right] = -\frac{2^{n+1}}{2}\frac{\Pi(n+1)}{n+1}\frac{x}{n}\frac{J_{n+1}(x)}{x^{n+1}},$$

$$\therefore \frac{d}{dx}\left[V_n(x)\right] = -\frac{x}{2n}V_{n+1}(x) = W_n(x).^{***}$$

---

* Gray, Mathews, and Macrobert, (7), page

** Tables of Spherical Bessel Functions, (12), page xlii.

*** $W_n(x)$ is defined by formula (a). Its values are given in Table 4 and Curve 3 of Appendix V.

$$\frac{x_\nu}{\rho^\nu(x)} \qquad \frac{x_\nu}{J^\nu(x)}$$

$$= -Y^\nu(x) + J^\nu(x)\left\{.1159\ldots + \frac{2}{\pi i}\right\} \cdot$$

$$\rho^\nu(x) = J^\nu(x)\left\{-\log\frac{x}{2} - \kappa + \frac{\pi i}{2}\right\}$$

$$Y^0(x) = J^{-1}(x) + \frac{4n(n+1)}{x_\nu Y^{n+1}(x)} \cdot$$

$$Y^{0-1}(x) = \left[1 - \frac{4n(n+1)}{x_\nu^2}\right] Y^{n+1}(x) - \frac{4(n+1)(n+2)}{x_\nu^2 Y^{n+2}(x)} \cdot$$

$$Y^0(x)\left[J^{-1}(x) - \frac{4n(n+1)}{x_\nu J^{n+1}(x)}\right] J^{-n}(x) = 1 \cdot$$

$$\frac{d}{dx}\left(\frac{x_\nu}{J^\nu(x)}\right) = -\frac{x_\nu}{J^{n+1}(x)} \cdot$$

$$\therefore \frac{d}{dx}\left(\sum_\nu \Pi(n) J^n(x)\right) = -\frac{x_\nu}{\sum_\nu \Pi(n) J^{n+1}(x)} \cdot$$

$$[n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots]$$

$$\therefore \frac{d}{dx}\left[Y^\nu(x)\right] = -\frac{x_\nu}{\sum_{\nu+1} \Pi(n+1)} \frac{n}{x} \frac{x_{\nu+1}}{J^{n+1}(x)} \cdot$$

$$\therefore \frac{d}{dx}\left[Y^\nu(x)\right] = -\frac{x_\nu}{2\nu} Y^{\nu+1}(x) = M^\nu(x) \cdot \quad ***$$

$Y^\nu(x)$ is defined by formula (5). The values are given in Table 4 and Diagram 3 of Appendix V.

b) $$\frac{d^n}{d(x^2)^n}\left[\frac{J_p(x)}{x^p}\right] = \left(-\frac{1}{2}\right)^n \frac{J_{p+n}(x)}{x^{p+n}} .$$ *

And using formula (b),

c) $$\frac{d}{d(x^2)}\left[V_n(x)\right] = -\frac{1}{4}\frac{V_{n+1}(x)}{n} .$$

d) $$\Lambda_{n+\frac{1}{2}}(x) = \frac{2^{2n+1}(-1)^n \Gamma\left(n+\frac{3}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{d^n}{d(x^2)^n}\left(\frac{\sin x}{x}\right)$$ **

where n is a positive integer.

---

* Stratton (1), page 42.

** Gray, Mathews, and MacRobert (7), page 38.

$$\frac{q_\nu}{q(x_s)}\left[V^\nu(x)\right] = -\frac{4}{\nu}\frac{\bar{V}^{\nu+1}(x)}{}\,.$$

$$\bar{V}^{\nu+\frac{1}{2}}(x) = \frac{(-1)^{\nu+1}\,L(\nu+\frac{1}{2})\gamma_\nu^{\nu+1}}{L(\frac{1}{2})\,q(x_s)^\nu}\left(\frac{\sin\gamma x}{x}\right)^{\nu+1}$$

## Appendix III

### Solution of Examples for Chapter III

III.1 The following distant field pattern amplitudes were obtained by substituting the corresponding known aperture amplitude distributions in the basic correlating integral equation (2.34) and carrying out the direct integration.

| Aperture Amplitude Distribution | Distant Field Pattern |
|---|---|
| a) $I(\hat{\rho},a) = H_o e^{-\beta^2\left(\frac{\pi\hat{\rho}}{2a}\right)^2}$ | $E(x) = 2H_o\pi a^2 e^{-\left(\frac{\beta\pi}{2}\right)^2}\left[\frac{J_1(x)}{x} + \sum_{n=2}^{\infty} 2^{n-1}\left(\frac{\beta\pi}{2}\right)^{2(n-1)} \frac{J_n(x)}{x^n}\right]$ |
| b) $I(\hat{\rho},a) = H_o(a^2 - \hat{\rho}^2)$ | $E(x) = 4H_o\pi a^4 \frac{J_2(x)}{x^2}$ |
| c) $I(\hat{\rho},a) = H_o$ | $E(x) = 2H_o\pi a^2 \frac{J_1(x)}{x}$ |
| d) $I(\hat{\rho},a) = H_o(a^2 - \hat{\rho}^2)^{\frac{1}{2}}$ | $E(x) = 2H_o\pi a^3\left[\frac{\sin x - x\cos x}{x^3}\right]$ |

( $G = 2\pi$ in the above examples.)

These known results will be used to check the method of solution which is outlined in Chapter III. Starting with the known distant field pattern, the corresponding aperture amplitude will be known to result.

III.2 Example (a):

The distant field pattern of Example (a),

$$E(x) = 2H_o\pi a^2 e^{-\left(\frac{\beta\pi}{2}\right)^2}\left[\frac{J_1(x)}{x} + \sum_{n=2}^{\infty} 2^{n-1}\left(\frac{\beta\pi}{2}\right)^{2(n-1)} \frac{J_n(x)}{x^n}\right],$$

may be written as the following summation of V functions:

$$E(x) = \frac{Ga^2}{2}\sum_{n=0}^{\infty} I_{\frac{n+1}{2}}\, V_{\frac{n+1}{2}}(x) \,. \qquad (3.32)$$

The equivalent pattern thus expressed is:

$$E(x) = H_o\pi a^2 e^{-\left(\frac{\beta\pi}{2}\right)^2}\left[\sum_{n=1}^{\infty}\left(\frac{\beta\pi}{2}\right)^{2(n-1)} \frac{V_n(x)}{(n-1)!}\right].$$

$$E(x) = H_o \pi a_s e^{-\left(\frac{\pi}{2}\right)^2} \left[ \sum_{n=1}^{\infty} \left(\frac{x}{2}\right)^{s(n-1)} \frac{V^n(x)}{(n-1)!} \right] .$$

$$I(\hat{\rho},\omega) = \frac{\pi}{2} \sum_{n=0}^{\infty} I_{n+1}^{\pi} V_{n+1}^{\pi}(x) .$$

$$E(x) = \sum H_o \pi a_s e^{-\left(\frac{\pi}{2}\right)^2} \left[ \frac{\pi}{2} I_1(x) + \sum_{n=2}^{\infty} \left(\frac{\pi}{2}\right)^{n-1} \frac{V_n(x)}{n!} \right] .$$

$$I(\hat{\rho},\omega) = H_o \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \qquad E(x) = \sum H_o \pi a_s \left[ \frac{J_1(x)}{x} \right]$$

$$I(\hat{\rho},\omega) = H_o \qquad E(x) = \sum H_o \pi a_s \frac{J_1(x)}{x}$$

$$I(\hat{\rho},\omega) = H_o \left(\omega_c - \hat{\rho}^2\right) \qquad E(x) = 4 H_o \pi a_s \frac{J_2(x)}{x^2}$$

$$I(\hat{\rho},\omega) = H_o \left(\omega_c - \hat{\rho}^2\right)^{\frac{1}{2}} \qquad E(x) = \sum H_o \pi a_s \left[ \frac{J_1(x)}{x} + \sum_{n=2}^{\infty} 2^{n-1} \left(\frac{\pi}{2}\right)^{n-1} \frac{J_n(x)}{x^n} \right]$$

By direct term to term correspondence, it is easily seen that $I_{\frac{1}{2}}$, $I_{\frac{3}{2}}$, etc. must be $\equiv 0$, and that

$$\frac{Ga^2}{2} I_n V_n(x) = H_o \pi a^2 e^{-\left(\frac{\beta\pi}{2}\right)^2} \left(\frac{\beta\pi}{2}\right)^{2(n-1)} \frac{V_n(x)}{(n-1)!} ,$$

for $n = 1, 2, 3, \cdots$.

Therefore,

$$I_n = \frac{2H_o \pi e^{-\left(\frac{\beta\pi}{2}\right)^2}}{G(n-1)!} \left(\frac{\beta\pi}{2}\right)^{2(n-1)} = \left[\frac{4H_o e^{-\left(\frac{\beta\pi}{2}\right)^2}}{G(n-1)!}\right]\left(\frac{\pi}{2}\right)\left(\frac{\beta\pi}{2}\right)^{2(n-1)} .$$

Call $\frac{4}{G} H_o e^{-\left(\frac{\beta\pi}{2}\right)^2} = \lambda$, then

$$I_n = \frac{\lambda}{(n-1)!} \frac{\pi}{2} \left(\frac{\pi}{2}\beta\right)^{2(n-1)} ,$$

and thus:

$$I_1 = \lambda \frac{\pi}{2} ,$$

$$I_2 = \lambda \frac{\pi}{2} \left(\frac{\pi}{2}\beta\right)^2 ,$$

$$I_3 = \frac{\lambda}{2!} \frac{\pi}{2} \left(\frac{\pi\beta}{2}\right)^4 , \text{ etc.}$$

Using the development of $I(\{1 - z^2\})$ as given in equation (3.10):

$$\sum_{n=0}^{\infty} \frac{I_{n+1}}{0!} = \frac{\lambda}{0!}\frac{\pi}{2} + \frac{\lambda}{1!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^2 + \frac{\lambda}{2!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^4 + \cdots = \frac{\lambda}{0!}\frac{\pi}{2} e^{\left(\frac{\pi}{2}\beta\right)^2} ,$$

$$\sum_{n=1}^{\infty} \frac{n I_{n+1}}{1!} = \frac{\lambda}{1!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^2 + \frac{\lambda}{2!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^2\left(\frac{\pi}{2}\beta\right)^2 + \frac{\lambda}{3!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^2\left(\frac{\pi}{2}\beta\right)^4 + \cdots$$

$$= \frac{\lambda}{1!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^2 e^{\left(\frac{\pi}{2}\beta\right)^2} ,$$

and similarly,

$$\sum_{n=m}^{\infty} \frac{n(n-1)\cdots(n-m+1)}{m!} I_{n+1} = \frac{\lambda}{m!}\frac{\pi}{2}\left(\frac{\pi}{2}\beta\right)^{2m} e^{\left(\frac{\pi}{2}\beta\right)^2} .$$

The coefficients of the powers of $z^{2n}$ are now known. Hence, $I(\{1 - z^2\})$ may be written:

$$\left\{ \quad \right\}$$



$$\sum_{\nu=0}^{\infty} \frac{\nu!}{\nu(\nu-1)\cdots(\nu-\nu+1)} I^{\nu+1} = \frac{\nu!}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right) \ominus_{\tau\nu} \left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau^{\nu}} .$$

$$= \frac{11}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right) \ominus \left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau} ,$$

$$\sum_{\nu=1}^{\infty} \frac{11}{\nu} I^{\nu+1} = \frac{11}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right) + \frac{11}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right)\left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau} + \frac{31}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right)\left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau}\left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau} + \cdots$$

$$\sum_{\nu=0}^{\infty} \frac{01}{\nu+1} I^{\nu+1} = \frac{01}{\sqrt{\pi}} \frac{\tau}{\pi} + \frac{11}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right) + \frac{11}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{a}\right)_{4} + \cdots = \frac{01}{\sqrt{\pi}} \frac{\tau}{\pi} \ominus \left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau} ,$$

$$\left\{ \quad \right\}_{\tau}$$

$$\tau = \frac{\tau 1}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{a}\right)_{4} ,$$

$$\tau = \sqrt{\pi} \left(\frac{\tau}{\pi}\mathrm{b}\right)_{\tau} ,$$

$$\mathrm{I} = \sqrt{\pi}^{\tau} ,$$

$$I^{\nu} = \frac{(\nu-1)!}{\sqrt{\pi}} \frac{\tau}{\pi} \left(\frac{\tau}{\pi}\mathrm{a}\right)_{\tau(\nu-1)} ,$$

$$\frac{\mathrm{G}}{4} \mathrm{H}^{\mathrm{o}} \mathrm{C}_{-\left(\frac{\mathrm{v}}{\pi}\right)^{\tau}\gamma}$$

$$I^{\nu} = \frac{\tau \mathrm{H}^{\mathrm{o}} \mathrm{u} \in}{\frac{(\nu-1)!}{\tau}} \left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau(\nu-1)}^{-\left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau}} = \left[\frac{\tau(\nu-1)!}{\tau+\mathrm{H}^{\mathrm{o}} \in}\right]^{-\left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau}} \left(\frac{\tau}{\mathrm{u}}\right)\left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau(\nu-1)} .$$

$$= \cdots$$

$$\frac{\tau}{\partial \mathrm{v}_{\mathrm{o}}} \mathrm{I}^{\nu} \Lambda^{\nu}(\mathrm{x}) = \mathrm{H}^{\mathrm{o}} \mathrm{u} \mathrm{v}_{\mathrm{o}} \epsilon^{-\left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau}} \left(\frac{\tau}{\partial \mathrm{u}}\right)_{\tau(\nu-1)} \frac{(\nu-1)!}{\Lambda^{\nu}(\mathrm{x})} .$$

$$= \equiv$$

$$\frac{\tau}{\tau}$$

$$I(\{1-z^2\}) = \lambda\frac{\pi}{2}e^{\left(\frac{\pi}{2}\beta\right)^2}\left[1-\frac{z^2}{1!}\left(\frac{\pi}{2}\beta\right)^2+\frac{z^4}{2!}\left(\frac{\pi}{2}\beta\right)^4+\cdots\right] = \lambda\frac{\pi}{2}e^{\left(\frac{\pi}{2}\beta\right)^2}\left(e^{-\left(\frac{\beta\pi z}{2}\right)^2}\right).$$

Substituting the value of $\lambda$:

$$I(\{1-z^2\}) = \frac{4H_0}{G}e^{-\left(\frac{\beta\pi}{2}\right)^2}e^{\left(\frac{\beta\pi}{2}\right)^2}\frac{\pi}{2}e^{-\left(\frac{\beta\pi z}{2}\right)^2}$$

$$= \frac{4H_0}{G}\frac{\pi}{2}e^{-\beta^2\left[\frac{\pi}{2}z\right]^2} = \frac{2H_0\pi}{G}e^{-\beta^2\left[\frac{\pi\hat{\rho}}{2a}\right]^2} = H_0 e^{-\beta^2\left[\frac{\pi\hat{\rho}}{2a}\right]^2},$$

which is the known aperture amplitude distribution for this example.

III.3 Example (b):

$$E(x) = 4H_0\pi a^4\frac{J_2(x)}{x^2} = H_0\pi a^4 V_2(x).$$

Since only $V_2(x)$ occurs, all $I_{\frac{n+1}{2}}$ except $I_2$ must be $\equiv 0$. Therefore,

$$H_0\pi a^4 V_2(x) = \frac{Ga^2}{2}I_2 V_2(x),$$

and $I_2 = \frac{2H_0\pi a^2}{G}$, all other $I \equiv 0$. Thus

$$I(\{1-z^2\}) = I_2(1-z^2)^1 = \frac{2H_0\pi a^2}{G}(1-z^2)$$

$$= \frac{2H_0\pi}{G}(a^2-\hat{\rho}^2) = H_0(a^2-\hat{\rho}^2),$$

which is the correct form for this example.

III.4 Example (c):

$$E(x) = 2H_0\pi a^2\frac{J_1(x)}{x} = H_0\pi a^2 V_1(x).$$

Since only $V_1(x)$ occurs, all $I_{\frac{n+1}{2}}$ except $I_1$ must be $\equiv 0$.

Therefore, $H_0\pi a^2 V_1(x) = \frac{Ga^2}{2}I_1 V_1(x)$,

$$H^{\circ}\Pi_{\nu}\Lambda^{1}(x) = \frac{\nu}{\nu+1}\Gamma^{1}\Lambda^{1}(x) \; ;$$

$$E(x) = 2H^{\circ}\Pi_{\nu}\frac{x_{\nu}}{\Gamma^{\nu}(x)} = H^{\circ}\Pi_{\nu}\Lambda^{1}_{\nu}(x) \; .$$

$$= \frac{2}{2H^{\circ}\Pi}\left(\hat{\sigma}_{\nu} - \hat{b}_{\nu}\right) = H^{\circ}\left(\hat{\sigma}_{\nu} - \hat{b}_{\nu}\right) \; ,$$

$$I((1-\xi_{\nu})) = \Gamma^{\nu}(1-\xi_{\nu}) = \frac{2}{2H^{\circ}\Pi_{\nu}}(1-\xi_{\nu})$$

$$\Gamma_{\nu} = \frac{2}{2H^{\circ}\Pi_{\nu}} \equiv$$

$$H^{\circ}\Pi_{\nu}\Lambda^{\nu}_{\nu}(x) = \frac{\nu}{\Gamma_{\nu}}\Gamma^{\circ}\Lambda^{\nu}_{\nu}(x) \; ,$$

$$\equiv \frac{\nu}{\nu+1} \qquad \nu$$

$$E(x) = 4H^{\circ}\Pi_{\nu}\varphi_{\nu}\frac{x_{\nu}}{\Gamma^{\nu}(x)} = H^{\circ}\Pi_{\nu}\Lambda^{\nu}_{\nu}(x) \; .$$

$$= \frac{2}{4H^{\circ}}\Pi_{\nu}\frac{2}{-\theta_{\nu}\left[\Pi\hat{x}\right]_{\nu}} = \frac{2}{2H^{\circ}\Pi}\frac{2}{-\theta_{\nu}\left[\Pi\hat{\theta}\right]_{\nu}} = H^{\circ}\frac{2}{-\theta_{\nu}\left[\Pi\hat{\theta}\right]_{\nu}} \; .$$

$$I((1-\xi_{\nu})) = \frac{2}{4H^{\circ}}\frac{2}{-\left(\frac{\pi}{2}\right)_{\nu}}\frac{\nu}{\left(\frac{\pi}{2}\right)_{\nu}}\Pi\frac{2}{-\left(\frac{\pi}{2\Pi x}\right)_{\nu}}$$

$$I((1-\xi_{\nu})) = \sqrt{\Pi}\frac{2}{\left(\frac{\pi}{2}\theta\right)_{\nu}}\left[1 - \frac{11}{\xi_{\nu}}\left(\frac{\pi}{2}\theta\right)_{\nu} + \frac{\nu i}{\xi_{\nu}}\left(\frac{\pi}{2}\theta\right)_{\nu} + \cdots\right] = \sqrt{\Pi}\frac{2}{\left(\frac{\pi}{2}\theta\right)_{\nu}}\left(\frac{2}{-\left(\frac{\pi}{2\Pi x}\right)_{\nu}}\right) \; .$$

and $I_1 = \frac{2H_0\pi}{G}$, all other $I \equiv 0$. Hence,

$$I(\{1-z^2\}) = I_1(1-z^2)^0 = I_1 = \frac{2H_0\pi}{G} = H_0,$$

which is the correct distribution.

III.5 Example (d):

$$E(x) = 2H_0\pi a^3\left[\frac{\sin x - x\cos x}{x^3}\right].$$

This pattern is of great interest because it first indicated the existence of $\Lambda$ and hence of $V$ functions to half-integer orders. The pattern may be expressed as follows:

$$E(x) = \frac{H_0\pi^{\frac{3}{2}}2^{\frac{1}{2}}a^3}{x^{\frac{3}{2}}}\left\{\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(\frac{\sin x - x\cos x}{x}\right)\right\}.$$

Since $J_{\frac{3}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\left(\frac{\sin x - x\cos x}{x}\right)$,

the pattern may be written as

$$E(x) = H_0\pi^{\frac{3}{2}}2^{\frac{1}{2}}a^3\frac{J_{\frac{3}{2}}(x)}{x^{\frac{3}{2}}}.$$

Now by equations (3.18) and (3.21),

$$V_{\frac{3}{2}}(x) = \frac{\Pi(\frac{3}{2})}{\frac{3}{2}}\frac{J_{\frac{3}{2}}(x)}{\left(\frac{x}{2}\right)^{\frac{3}{2}}} = \pi^{\frac{1}{2}}2^{\frac{1}{2}}\frac{J_{\frac{3}{2}}(x)}{x^{\frac{3}{2}}},$$

and the distant field pattern of example (d) becomes

$$E(x) = H_0\pi a^3 V_{\frac{3}{2}}(x).$$

Since the only term appearing in $E(x)$ is that in $V_{\frac{3}{2}}(x)$, all $I_{\frac{n+1}{2}}$ except $I_{\frac{3}{2}}$ must be $\equiv 0$.

Therefore, $$\frac{Ga^2}{2}I_{\frac{3}{2}}V_{\frac{3}{2}}(x) = H_0\pi a^3 V_{\frac{3}{2}}(x),$$

or $I_{\frac{3}{2}} = \frac{2H_0\pi a}{G}$, all other $I \equiv 0$.

$$\frac{\partial}{\partial x}\left(\frac{2H_0\pi y}{x}\right) =$$

$$\frac{\partial}{\partial x}\left(\frac{2H_0\pi y}{x}\right) = H_0\pi a_3 J_{\frac{1}{2}}(x) = H_0\pi a_3 \sqrt{\frac{2}{\pi x}}\sin x \,.$$

$$= \qquad\qquad \frac{\nu+1}{x}$$

$$E(x) = H_0\pi a_3 \Lambda_{\frac{1}{2}}(x).$$

$$\Lambda_{\frac{1}{2}}(x) = \frac{1}{\Gamma(\frac{1}{2})}\frac{\left(\frac{x}{2}\right)^{\frac{1}{2}}}{x^{\frac{1}{2}}} = \pi^{-\frac{1}{2}}\frac{J_{\frac{1}{2}}(x)}{x^{\frac{1}{2}}},$$

$$E(x) = H_0\pi \frac{x}{x^{\frac{1}{2}}} a_3 J_{\frac{1}{2}}(x) \,.$$

$$J_{\frac{1}{2}}(x) = \left(\frac{\pi x}{2}\right)^{\frac{1}{2}}\left(\frac{\sin x}{x} - \cos x\right),$$

$$E(x) = H_0\pi \frac{x}{x^{\frac{1}{2}}} a_3 \left\{\left(\frac{\pi x}{2}\right)^{\frac{1}{2}}\left(\frac{\sin x}{x} - \cos x\right)\right\}.$$

$$\gamma \qquad \Lambda$$

$$E(x) = 2H_0\pi a_3 \left[\frac{\sin x}{x^2} - \frac{x\cos x}{x^2}\right].$$

$$I(1-\tfrac{1}{2}) = \Gamma(1-\tfrac{1}{2})_0 = \Gamma' = \frac{2H_0\pi}{\zeta} = H_0 \,,$$

$$= \frac{2H_0\pi}{\pi^{\frac{1}{2}}}$$

$$\text{Now } I(\{1-s^2\}) = H_0 a (1-z^2)^{\frac{1}{2}} = H_0 a \left(\frac{a^2-\hat{\rho}^2}{a^2}\right)^{\frac{1}{2}}$$
$$= H_0 (a^2-\hat{\rho}^2)^{\frac{1}{2}}.$$

which again is the correct answer for the aperture distribution.

## Appendix IV

### Solution of Examples for Chapter IV

IV.1 To illustrate the method of solution proposed in Chapter IV, let $E(x) = \frac{Ga^2}{2}\left[I_1 V_1(x) + I_3 V_3(x)\right]$ where $I_1 = 1$, $I_3 = 3$, and $Ga^2 = 2$. Therefore, $E(x) = V_1(x) + 3V_3(x)$. Since, by (3.16) and (3.21),

$$V_1 = \Lambda_1 = 1 - \frac{x^2}{8} + \frac{x^4}{12\cdot 16} - \frac{x^6}{6\cdot 24\cdot 64} + \cdots,$$

$$V_3 = \frac{\Lambda_3}{3} = \frac{1}{3} - \frac{x^2}{48} + \frac{x^4}{120\cdot 16} - \frac{x^6}{3\cdot 720\cdot 64} + \cdots.$$

$E(x)$ may be written:

$$E(x) = V_1(x) + 3V_3(x)$$

$$= 2 - x^2\left[\frac{1}{8} + \frac{1}{16}\right] + x^4\left[\frac{1}{12\cdot 16} + \frac{1}{40\cdot 16}\right] - x^6\left[\frac{1}{6\cdot 24\cdot 64} + \frac{1}{120\cdot 64}\right] + \cdots$$

$$= 2 - \frac{3}{16}x^2 + \frac{13}{120\cdot 16}x^4 - \frac{6}{120\cdot 64}x^6 + \cdots.$$

IV.2 Following the method of Chapter IV, we will multiply $E(x)$ by $(e^{-ixz}\,dx)$ and integrate over the limits $0 \sim \infty$ by use of formula (4.30), $\int_0^\infty e^{-ixz} x^{2p}\,dx = \frac{(2p)!}{[iz]^{2p+1}}$, as follows:

$$\int_0^\infty e^{-ixz} E(x)\,dx = \int_0^\infty e^{-ixz}\left[2 - \frac{3}{16}x^2 + \frac{13}{120\cdot 16}x^4 - \frac{6}{120\cdot 64}x^6 + \cdots\right]dx$$

$$= \frac{2}{[iz]} - \frac{6}{16[iz]^3} + \frac{13}{80[iz]^5} - \frac{72}{128[iz]^7} + \cdots$$

IV.3 Let us consider the possibility that $V_1$, $V_2$, and $V_3$ occur in the arbitrary $E(x)$. Then by use of (4.36) we will set up three simultaneous equations in the unknowns $\Pi_1$, $\Pi_2$, and $\Pi_3$.

$$x=0: \quad 2 = \Pi_1\left[\frac{(\frac{1}{2})}{1!}\right] + \Pi_2\left[\frac{(\frac{3}{2})(\frac{1}{2})}{2!}\right] + \Pi_3\left[\frac{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})}{3!}\right]$$

$$x=1: \quad -\frac{3}{8} = \Pi_1\left[\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\right] + \Pi_2\left[\frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{3!}\right] + \Pi_3\left[\frac{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})}{4!}\right]$$

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\,dx$$

$$\int_0^\infty e^{-ix} E(x)\,dx = \int_0^\infty e^{-ix}\left[1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \cdots\right]dx$$

$$\int_0^\infty e^{-ix} x^p\,dx = \frac{\Gamma(p+1)}{(ip)!}$$

$$= \frac{\Gamma(1)}{(i)!} - \frac{6}{\Gamma(i)^3} + \frac{15}{8C\,\Gamma(i)^5} - \frac{72}{128\,\Gamma(i)^7} + \cdots$$

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\,dx$$

$$= \Gamma(1)\left[\frac{(i)^0}{0!}\right] + \Gamma(2)\left[\frac{(i)^1}{1!}\right] + \Gamma(3)\left[\frac{(i)^2}{2!}\right] + \Gamma(4)\left[\frac{(i)^3}{3!}\right]$$

$$= \Gamma(1)\left[\frac{(i)^0}{0!}\right] + \Gamma(2)\left[\frac{(i)^1}{1!}\right] + \Gamma(3)\left[\frac{(i)^2}{2!}\right] + \Gamma(4)\left[\frac{(i)^3}{3!}\right]$$

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1}\,dx$$

$$= \Gamma(1) + \Gamma(2) + \Gamma(3) + \Gamma(4)$$

$$= \Gamma(1)\left[\frac{(i)^0}{0!}\right] + \Gamma(2)\left[\frac{(i)^1}{1!}\right] + \Gamma(3)\left[\frac{(i)^2}{2!}\right] + \Gamma(4)\left[\frac{(i)^3}{3!}\right]$$

$$E(x) = V^1(x) + i\,V^3(x)$$

$$V^1 = V^1 = 1 - \frac{x^2}{2} + \frac{x^4}{12\cdot 16} - \frac{x^6}{6\cdot 24\cdot 64} + \cdots,$$

$$V^3 = \frac{V^3}{i} = x - \frac{x^3}{3} + \frac{x^5}{120\cdot 16} - \frac{x^7}{3\cdot 120\cdot 64} + \cdots.$$

$$= x - x^3\left[\frac{1}{8} + \frac{1}{16}\right] + x^5\left[\frac{1}{120\cdot 16} + \frac{1}{40\cdot 16}\right] - x^7\left[\frac{1}{6\cdot 24\cdot 64} + \frac{1}{120\cdot 64}\right] + \cdots$$

$$= x - \frac{1}{16}x^3 + \frac{1}{120\cdot 16}x^5 - \frac{1}{120\cdot 64}x^7 + \cdots.$$

$$k = 2: \quad \frac{13}{80} = \Pi_1\left[\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\right] + \Pi_2\left[\frac{(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{4!}\right] + \Pi_3\left[\frac{(\frac{5}{2})(\frac{3}{2})(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{5!}\right]$$

Simplifying the three equations becomes:

$k = 0$: $\quad 6\Pi_1 + 6\Pi_2 + 5\Pi_3 = 32,$

$k = 1$: $\quad 16\Pi_1 + 8\Pi_2 + 6\Pi_3 = 48,$

$k = 2$: $\quad 80\Pi_1 + 30\Pi_2 + 15\Pi_3 = 208.$

IV.4 Solving for the unknowns by the method of determinants:

$$\Pi_1 = \frac{\begin{vmatrix} 32 & 6 & 5 \\ 48 & 8 & 5 \\ 208 & 30 & 15 \end{vmatrix}}{\begin{vmatrix} 8 & 6 & 5 \\ 16 & 8 & 5 \\ 80 & 30 & 15 \end{vmatrix}} = \frac{16\cdot 2\cdot 5 \begin{vmatrix} 2 & 3 & 1 \\ 3 & 4 & 1 \\ 13 & 15 & 3 \end{vmatrix}}{8\cdot 2\cdot 5 \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 10 & 15 & 3 \end{vmatrix}} = \frac{2(-1)}{(-1)} = 2;$$

$$\Pi_2 = \frac{\begin{vmatrix} 8 & 32 & 5 \\ 16 & 48 & 5 \\ 80 & 208 & 15 \end{vmatrix}}{\begin{vmatrix} 8 & 6 & 5 \\ 16 & 8 & 5 \\ 80 & 30 & 15 \end{vmatrix}} = \frac{8\cdot 16\cdot 5 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 10 & 13 & 3 \end{vmatrix}}{8\cdot 2\cdot 5 \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 10 & 15 & 3 \end{vmatrix}} = \frac{8(0)}{-1} = 0;$$

$$\Pi_3 = \frac{\begin{vmatrix} 8 & 6 & 32 \\ 16 & 8 & 48 \\ 80 & 30 & 208 \end{vmatrix}}{\begin{vmatrix} 8 & 6 & 5 \\ 16 & 8 & 5 \\ 80 & 30 & 15 \end{vmatrix}} = \frac{8\cdot 2\cdot 16 \begin{vmatrix} 1 & 3 & 2 \\ 2 & 4 & 3 \\ 10 & 15 & 13 \end{vmatrix}}{8\cdot 2\cdot 5 \begin{vmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 10 & 15 & 3 \end{vmatrix}} = \frac{16(-1)}{5(-1)} = \frac{16}{5}.$$

IV.5 Now by (4.20), $I_n = \dfrac{[1\cdot 3\cdot 5\cdots(2n-1)]}{Ga^2 2^{n-1}\Gamma(n)}\Pi_n$, therefore

$$I_1 = \frac{\Pi_1}{2} = \frac{2}{2} = 1;$$

$$\Gamma' = \frac{\nu}{\Pi'} = \frac{\nu}{\nu} = 1 ;$$

$$u = \frac{\mathfrak{C}\gamma_\nu \Sigma_{\mu_\nu} L(\mu)}{\Gamma 1 \cdot 3 \cdot 5 \cdots (2\nu-1)\, \Gamma}\, \Pi^\nu$$

$$\Pi^3 = \frac{\begin{vmatrix} 80 & 30 & 12 \\ 16 & 8 & 2 \\ 8 & 6 & 2 \end{vmatrix}}{\begin{vmatrix} 80 & 30 & 208 \\ 16 & 8 & 44 \\ 8 & 6 & 32 \end{vmatrix}} = \frac{\begin{vmatrix} 10 & 12 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{8 \cdot 2 \cdot 16} = \frac{2\,(-1)}{16\,(-1)} = \frac{2^5}{16^2} .$$

$$\Pi^5 = \frac{\begin{vmatrix} 80 & 30 & 12 \\ 16 & 8 & 2 \\ 8 & 6 & 2 \end{vmatrix}}{\begin{vmatrix} 160 & 208 & 12 \\ 16 & 48 & 2 \\ 8 & 32 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 10 & 12 & 3 \\ 2 & 4 & 1 \\ 1 & 3 & 1 \end{vmatrix}}{8 \cdot 2 \cdot 2} = \frac{\begin{vmatrix} 10 & 12 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix}}{8 \cdot 16 \cdot 2} = \frac{-1}{8\,(0)} = 0 ;$$

$$\Pi^7 = \frac{\begin{vmatrix} 80 & 30 & 12 \\ 16 & 8 & 2 \\ 8 & 6 & 2 \end{vmatrix}}{\begin{vmatrix} 320 & 208 & 12 \\ 64 & 48 & 2 \\ 32 & 32 & 2 \end{vmatrix}} = \frac{\begin{vmatrix} 10 & 12 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & 1 \end{vmatrix}}{8 \cdot 2 \cdot 2} = \frac{\begin{vmatrix} 10 & 12 & 3 \\ 3 & 4 & 1 \\ 2 & 3 & 1 \end{vmatrix}}{16 \cdot 2 \cdot 2} = \frac{(-1)}{2\,(-1)} = \frac{1}{2} ;$$

$$= \Pi^1 + \Pi^3 + \Pi^5 = \Pi^7 =$$

$$= \Pi^1 + \Pi^3 + \Pi^5 = \Pi^7 =$$

$$= \Pi^1 + \Pi^3 + \Pi^5 = \Pi^7 =$$

$$= \frac{\mathfrak{C}\gamma_\nu}{\Gamma} = \Pi\left[\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\right] + \Pi^3\left[\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{5!}\right] + \Pi^5\left[\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{7!}\right]$$

$$I_2 = \frac{3\,\Pi_2}{2\cdot 2} = 0;$$

$$I_3 = \frac{3.5}{2\cdot 4\cdot 2}\Pi_3 = \frac{15}{16}\Pi_3 = \frac{15}{16}\cdot\frac{16}{5} = 3.$$

IV.6 The aperture distribution is thus $I(z) = 1 + 3(1-z^2)^2$, which is the result expected from the observations of Chapter III.

$$I_2 = \frac{3\,\Pi_2}{2\cdot 2} = 0;$$

$$I_3 = \frac{3\cdot 5}{2\cdot 4\cdot 2}\,\Pi_3 = \frac{15}{16}\,\Pi_3 = \frac{15}{16}\cdot\frac{16}{5} = 3.$$

$$I = x + \ldots + z_r^2$$

## Appendix V

### Curves and Tables


| | | |
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TABLE 1

VALUES OF $V_n(x)$

| x | $V_{\frac{1}{2}}$ | $V_1$ | $V_{\frac{3}{2}}$ | $V_2$ | $V_{\frac{5}{2}}$ | $V_3$ | $V_{\frac{7}{2}}$ | $V_4$ | $V_{\frac{9}{2}}$ | $V_5$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 2.0000 | 1.0000 | .6667 | .5000 | .4000 | .3333 | .286 | .2500 | .2222 | .2000 |
| 0.2 | | .9950 | | .4983 | | .3325 | | .2495 | | .1997 |
| 0.5 | | .9691 | | .4896 | | .3282 | | .2466 | | .1979 |
| 0.7 | | .9400 | | .4799 | | .3233 | | .2439 | | .1959 |
| 1.0 | 1.682 | .8801 | .61 | .4596 | .372 | .3130 | | .2378 | | .1918 |
| 1.5 | | .7439 | | .4126 | | .2857 | | .2232 | | .1820 |
| 2.0 | .910 | .5767 | .436 | .3528 | .296 | .2579 | .2276 | .2040 | .1860 | .1689 |
| 2.5 | | .3977 | | .2855 | | .2218 | | .1814 | | .1534 |
| 3.0 | .094 | .2260 | .228 | .2160 | .1985 | .1831 | .1686 | .1565 | .1460 | .1360 |
| 4.0 | -.376 | -.0330 | .058 | .0910 | .1034 | .1075 | .107 | .1054 | .1035 | .0991 |
| 5.0 | -.383 | -.1310 | -.038 | .0075 | .0327 | .0467 | .055 | .0601 | .0632 | .0642 |
| 6.0 | -.093 | -.0922 | -.0556 | -.0270 | -.0062 | .0085 | .0188 | .0265 | .0322 | .0358 |
| 7.0 | .188 | -.0013 | -.0268 | -.0246 | -.0164 | -.0078 | | .0063 | .0117 | .0159 |
| 8.0 | .247 | .0587 | .0084 | -.0071 | -.0104 | -.0091 | -.006 | -.0025 | .0011 | .0044 |
| 9.0 | .0915 | .0575 | .0236 | .0072 | -.0008 | -.0040 | -.0047 | -.0039 | -.0025 | -.0007 |
| 9.8 | | .0097 | | .0102 | | .0009 | | -.0022 | | -.0018 |
| 10.0 | -.109 | | .0157 | | .0047 | | -.0012 | | -.0022 | |

Note: This table is provided as an indication of values. For greater accuracy, use values of $\Lambda_n(x)$ and formula $V_n(x) = \frac{\Lambda_n(x)}{n}$.

TABLE 2

VALUES OF $\Lambda_n(x)$

| x | $\Lambda_1$ | $\Lambda_2$ | $\Lambda_3$ | $\Lambda_4$ | $\Lambda_5$ | $\Lambda_6$ | $\Lambda_7$ | $\Lambda_8$ | $\Lambda_9$ | $\Lambda_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.5 | .96907 | .97933 | .98447 | .98756 | .98963 | .99111 | .99221 | .99308 | .99377 | .99433 |
| 1.0 | .88010 | .91923 | .93904 | .95103 | .95907 | .96484 | .96918 | .97257 | .97528 | .97751 |
| 1.5 | .74392 | .82520 | .86704 | .89263 | .90493 | .92241 | .93184 | .93923 | .94517 | .95005 |
| 2.0 | .57627 | .70567 | .77366 | .81590 | .84476 | .86575 | .88172 | .89428 | .90442 | .91278 |
| 2.5 | .39768 | .57096 | .66540 | .72531 | .76684 | .79737 | .82079 | .83933 | .85438 | .86684 |
| 3.0 | .22604 | .43208 | .54944 | .62594 | .67996 | .72021 | .75140 | .77629 | .79663 | .81357 |
| 3.5 | .07850 | .29951 | .43300 | .52306 | .58813 | .63743 | .67611 | .70729 | .73296 | .75447 |
| 4.0 | -.03302 | .18206 | .32263 | .42169 | .49523 | .55224 | .59756 | .63454 | .66523 | .69117 |
| 4.5 | -.10269 | .08606 | .22371 | .32628 | .40520 | .46767 | .51831 | .56017 | .59535 | .62532 |
| 5.0 | -.13103 | .01490 | .14010 | .24037 | .32089 | .38648 | .44076 | .48634 | .52513 | .55851 |
| 5.5 | -.12416 | -.03103 | .07389 | .16648 | .24486 | .31094 | .36697 | .41489 | .45625 | .49227 |
| 6.0 | -.09223 | -.05397 | .02550 | .10597 | .17881 | .24280 | .29864 | .34741 | .39021 | .42796 |
| 6.5 | -.04734 | -.05821 | -.00618 | .05912 | .12363 | .18324 | .23703 | .28519 | .32823 | .36677 |
| 7.0 | -.00134 | -.04291 | -.02345 | .02524 | .07949 | .13285 | .18298 | .22913 | .27128 | .30965 |
| 7.5 | .03607 | -.03275 | -.02936 | .00289 | .04587 | .09169 | .13685 | .17982 | .22033 | .25733 |
| 8.0 | .05866 | -.01412 | -.02729 | -.00988 | .02177 | .05934 | .09862 | .13748 | .17486 | .21030 |
| 8.5 | .06426 | .00247 | -.02053 | -.01528 | .00581 | .03503 | .06794 | .10203 | .13591 | .16880 |
| 9.0 | .05451 | .01431 | -.01193 | -.01554 | -.00358 | .01772 | .04417 | .07315 | .10305 | .13289 |
| 9.5 | .03395 | .02020 | -.00366 | -.01269 | -.00801 | .00623 | .02649 | .05030 | .07598 | .10242 |
| 10.0 | .00869 | .02037 | .00280 | -.00843 | -.00899 | .00067 | .01398 | .03281 | .05423 | .07710 |

TABLE 5

POWERS OF $(1 - z^2)^n$

$(n = 1, 2, 3, \cdots, 19)$

| z | n = 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| .1 | .99 | .9801 | .9703 | .9606 | .9510 | .9415 | .9321 |
| .2 | .96 | .9216 | .8847 | .8493 | .8154 | .7828 | .7514 |
| .3 | .91 | .8281 | .7536 | .6857 | .6240 | .5679 | .5168 |
| .4 | .84 | .7056 | .5927 | .4979 | .4182 | .3513 | .2951 |
| .5 | .75 | .5625 | .4219 | .3164 | .2373 | .1780 | .1335 |
| .6 | .64 | .4096 | .2621 | .1678 | .1074 | .0687 | .0440 |
| .7 | .51 | .2601 | .1327 | .0677 | .0345 | .0176 | .0090 |
| .8 | .36 | .1296 | .0467 | .0168 | .0060 | .0022 | .0008 |
| .9 | .19 | .0361 | .0069 | .0013 | .0002 | .0000 | .0000 |
| 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

| z | n = 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| .1 | .9227 | .9135 | .9044 | .8953 | .8864 | .8775 | .8687 |
| .2 | .7214 | .6925 | .6648 | .6382 | .6127 | .5882 | .5647 |
| .3 | .4703 | .4279 | .3894 | .3544 | .3225 | .2935 | .2670 |
| .4 | .2479 | .2082 | .1749 | .1469 | .1234 | .1037 | .0871 |
| .5 | .1001 | .0751 | .0563 | .0422 | .0317 | .0238 | .0178 |
| .6 | .0261 | .0180 | .0115 | .0074 | .0047 | .0030 | .0019 |
| .7 | .0046 | .0023 | .0011 | .0006 | .0003 | .0001 | .0001 |
| .8 | .0003 | .0001 | .0000 | .0000 | .0000 | .0000 | .0000 |
| .9 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

| z | n = 15 | 16 | 17 | 18 | 19 |
|---|---|---|---|---|---|
| 0.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |
| .1 | .8601 | .8515 | .8429 | .8345 | .8262 |
| .2 | .5421 | .5204 | .4996 | .4796 | .4604 |
| .3 | .2430 | .2211 | .2012 | .1831 | .1666 |
| .4 | .0731 | .0614 | .0516 | .0434 | .0364 |
| .5 | .0134 | .0100 | .0075 | .0056 | .0042 |
| .6 | .0012 | .0008 | .0005 | .0003 | .0002 |
| .7 | .0000 | .0000 | .0000 | .0000 | .0000 |
| .8 | .0000 | .0000 | .0000 | .0000 | .0000 |
| .9 | .0000 | .0000 | .0000 | .0000 | .0000 |
| 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |

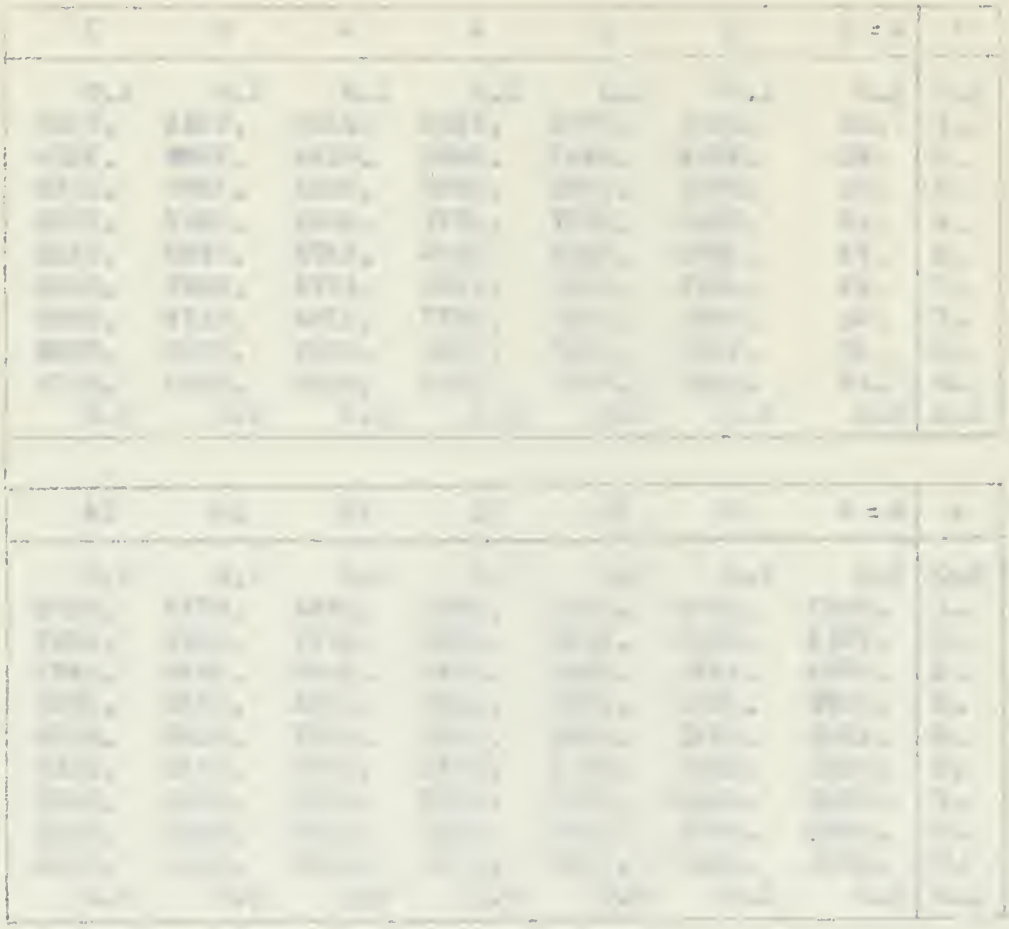
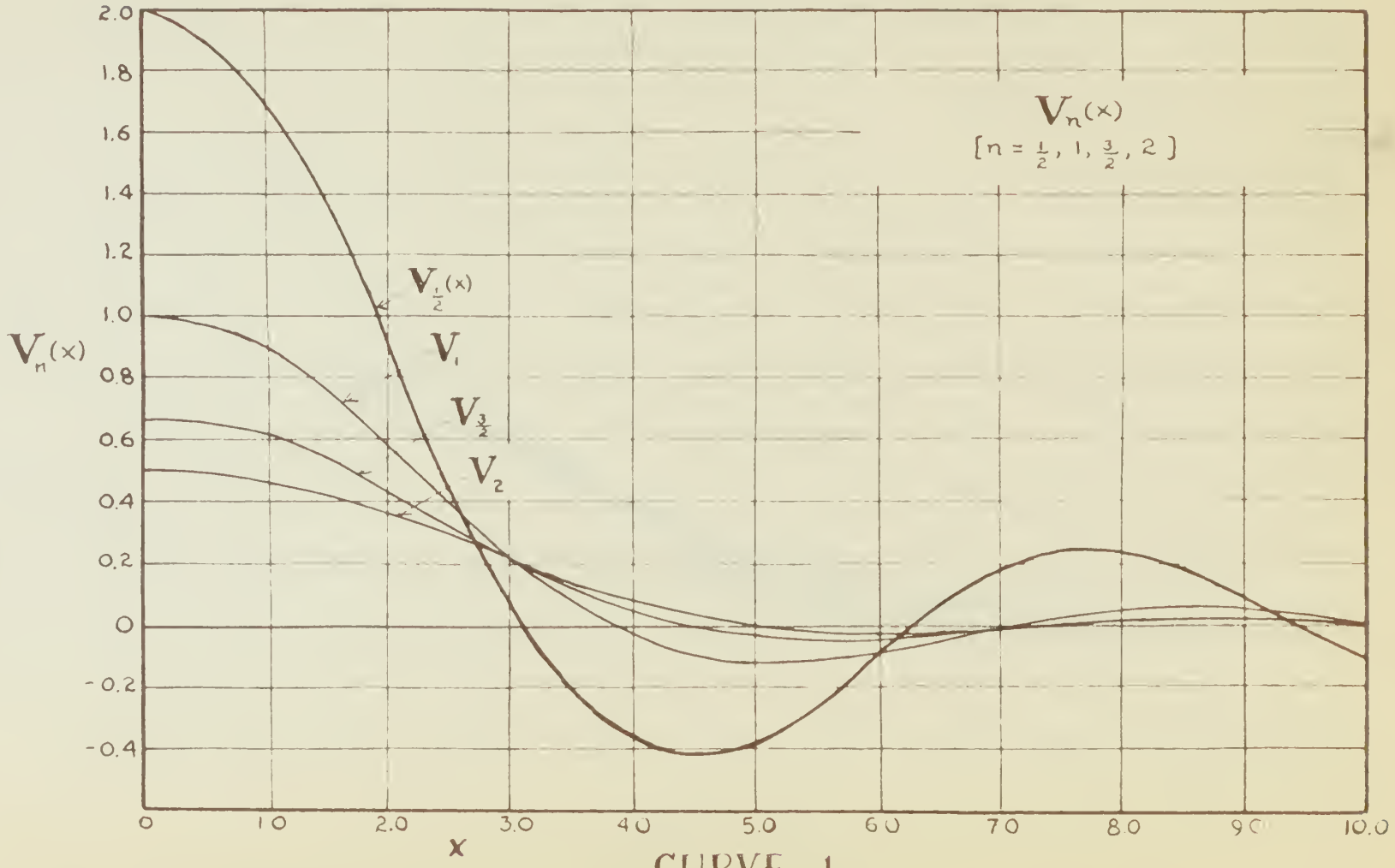


TABLE 4

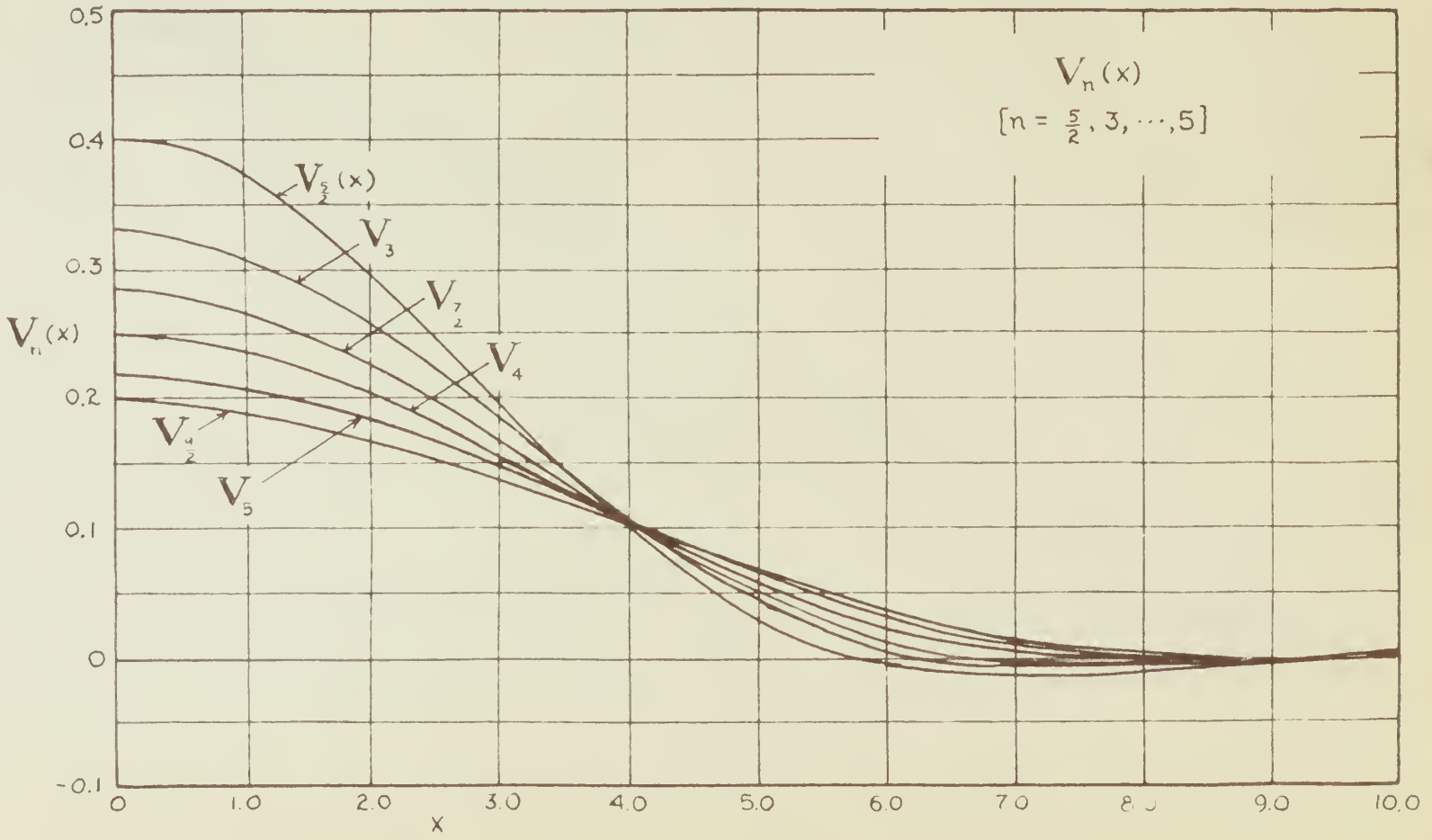
VALUES OF $W_n(x)$

| x | $W_{\frac{1}{2}}$ | $W_1$ | $W_{\frac{3}{2}}$ | $W_2$ | $W_{\frac{5}{2}}$ | $W_3$ | $W_{\frac{7}{2}}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 1 | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | | -.[illegible] | |
| 2 | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] |
| 3 | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] |
| 4 | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] |
| 5 | .[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] |
| 6 | .[illegible] | .[illegible] | .[illegible] | -.[illegible] | -.[illegible] | -.[illegible] | -.[illegible] |
| 7 | .[illegible] | .[illegible] | .[illegible] | .[illegible] | | -.[illegible] | -.[illegible] |
| 8 | -.[illegible] | .[illegible] | .[illegible] | .[illegible] | .[illegible] | .[illegible] | .[illegible] |
| 9 | -.[illegible] | -.[illegible] | .[illegible] | .[illegible] | .[illegible] | .[illegible] | .[illegible] |
| 10 | -.[illegible] | | | | | | |



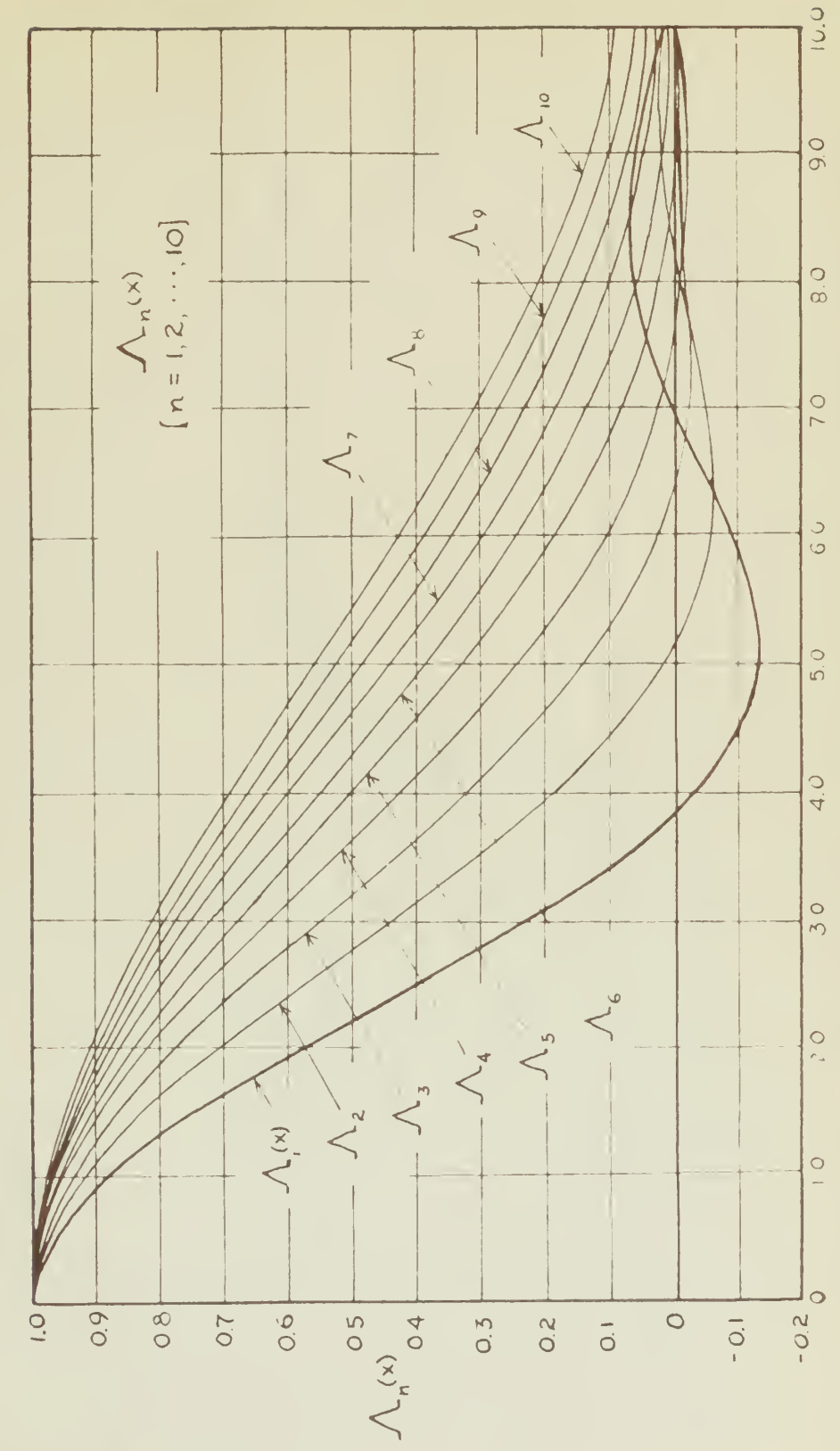

CURVE 1

PLOT OF $V_n(x)$
$[n = \frac{1}{2}, 1, \frac{3}{2}, 2]$



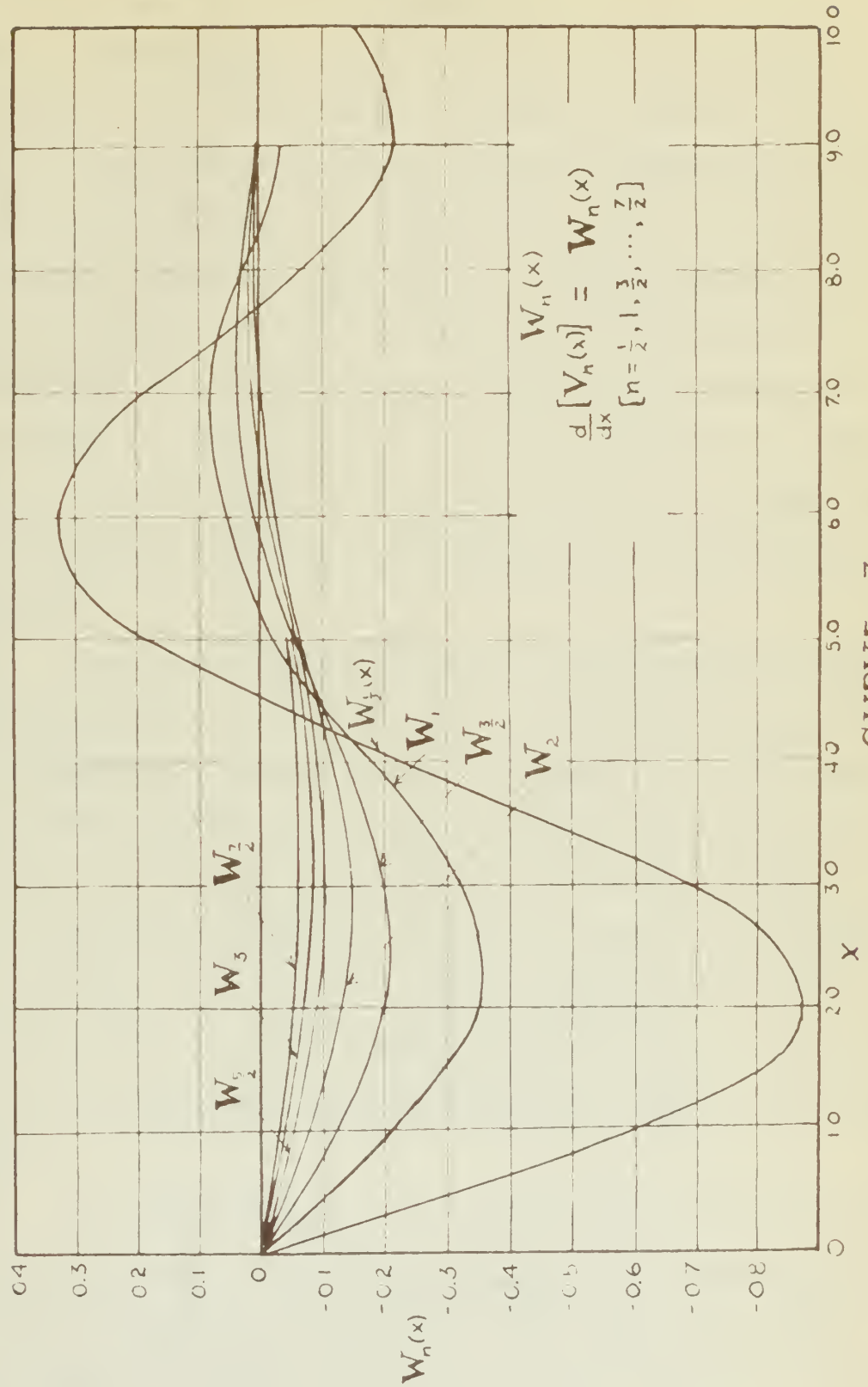

**CURVE 1a**

PLOT OF $V_n(x)$
$[n = \frac{5}{2}, 3, \cdots, 5]$




CURVE 2

PLOT OF $\Lambda_n(x)$
$[n = 1, 2, \cdots, 10]$




CURVE 3
PLOT OF $W_n(x)$
$[n = \frac{1}{2}, 1, \frac{3}{2}, \ldots, \frac{7}{2}]$

## Appendix VI

### Addendum

VI.1 While preparing a talk on the subject of this paper for presentation at the Naval Research Laboratory, the author was led to some very interesting developments. These developments are considered to be of such importance that they are here appended to this essay. The first is a comparison of the results of the proposed analytical methods with actual measured data. The second is the outline of a new method of distant field radiation pattern analysis which places the methods of solution previously presented in this paper on sound practical footing. Curves and tables illustrating this Appendix appear herein instead of in Appendix V so they may be available for ready reference.

VI.2 Unclassified experimental data was obtained from the Applied Physics Laboratory for the express purpose of checking the correspondence of the theoretical methods with practical results. This experimental data is presented in normalized form in Curves (4a) and (4b) and in Table (5). It was first necessary to express the actual measured distant field pattern (Curve 4a) analytically. Once this was done, it was desired that both proposed methods be applied. The resultant expressions for aperture illumination compared to the actual measured illumination would then graphically evaluate the worth of the proposed theory.

VI.3 The method of Chapter III was tested first. By visual inspection, it was determined that $V_0(x)$ and $V_{12}(x)$ could reasonably well satisfy the requirements of expression. This rather casual decision could certainly be improved if one were willing to tolerate the mathematical complications of more terms. Check points at $x = 0$ and $x = 5$

were used to solve for the coefficients of $V_2(x)$ and $V_{12}(x)$. The following simultaneous equations were solved:

$$AV_2(x) + BV_{12}(x) = E(x); \quad x = 0, .5.$$

Values of A = 1.08, B = 5.5 were obtained. This expression was then tabulated (Table 5A) and plotted (Curve 4a). While not an exact representation of E(x), it is believed that the essential character of E(x) is portrayed well enough for demonstration purposes. The method of Chapter III indicates an aperture illumination of

$$I(z) = \frac{2}{Ca^2}\left[1.08(1 - z^2) + 5.5(1 - z^2)''\right].$$

This aperture illumination is tabulated in (Table 5B) and plotted on (Curve 4b). The correspondence of theory and practice is clearly indicated. This comparison is considered very good, inasmuch as the theory does not take into account such things as feed shadow and power lost in radiation. The mathematical connection between E(x) as expressed in V functions and the aperture illumination of this method is completely rigid. The only mathematical approximation in this entire method is then the expression of E(x). Certainly, the engineer faced with the problem of designing an aperture illumination to produce a given beam would do well to first examine such theory.

VI.4 To test the method of Chapter IV in as simple a fashion as possible, an exponential approximation of E(x) was used.

$$E(x) \sim e^{-\frac{x^2}{20}}$$

This function is also tabulated in (Table 5A) and plotted on (Curve 4a). The correspondence is not as good as in the first case, but it was also desirable to observe the effects of approximation. The exponential was

TABLE 5

COMPARISON OF EXPERIMENTAL DATA
WITH ANALYTICAL METHODS

(A) Distant Field Pattern (Curve 4 a)

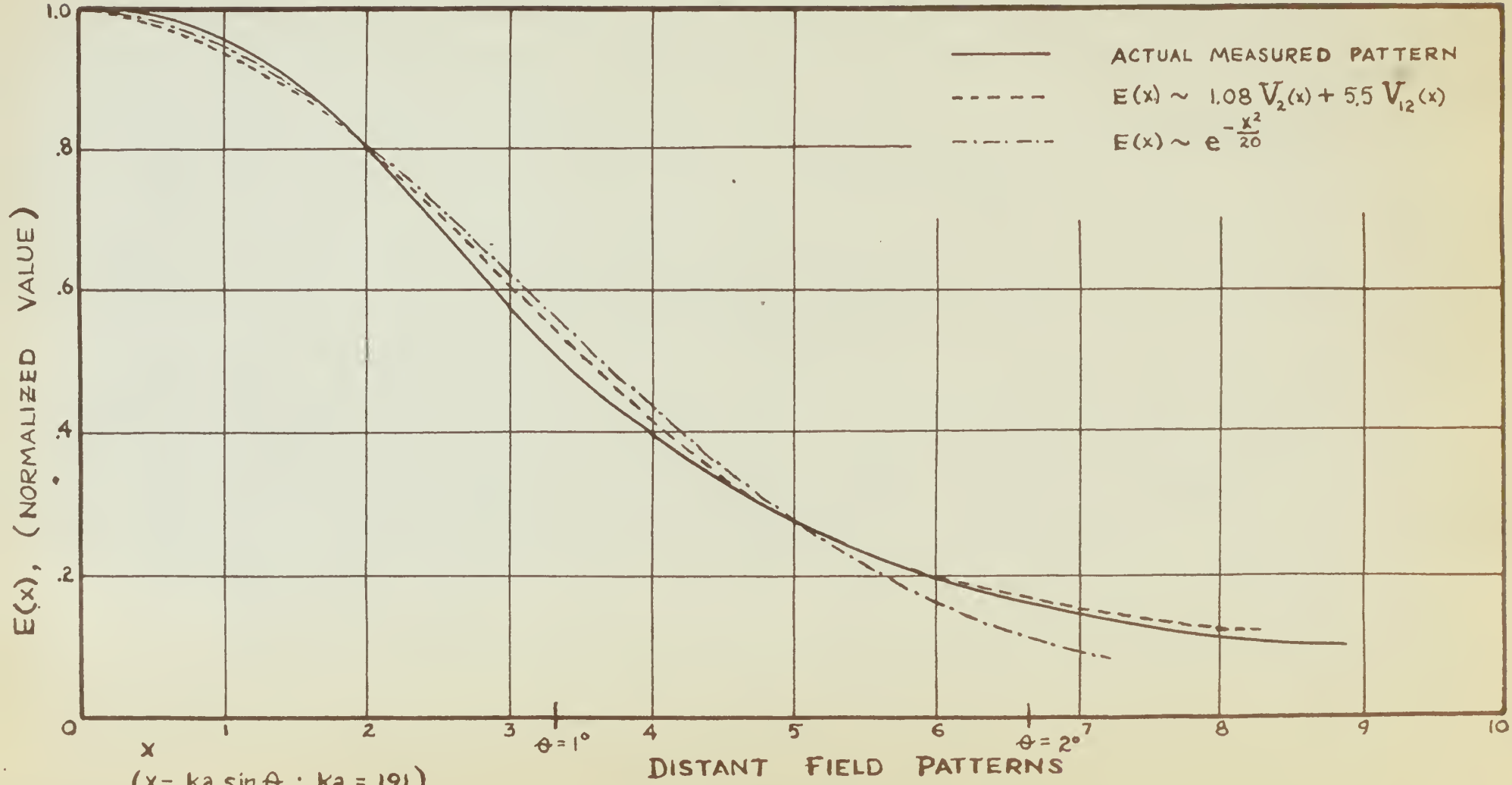
| x | E(x) Measured * | $E(x) \sim 1.08V_2 + 5.5V_{12}$ | $E(x) \sim e^{-\frac{x^2}{20}}$ |
|---|---|---|---|
| 0 | 1.00 | 1.00 | 1.00 |
| 1 | 0.96 | 0.95 | 0.95 |
| 2 | 0.81 | 0.81 | 0.82 |
| 3 | 0.57 | 0.62 | 0.64 |
| 4 | 0.39 | 0.43 | 0.45 |
| 5 | 0.28 | 0.28 | 0.28 |
| 6 | 0.20 | 0.20 | 0.17 |
| 7 | 0.15 | 0.15 | 0.09 |
| 8 | 0.11 | 0.13 | |
| 9 | 0.10 | | |
| 10 | 0.09 | | |

(B) Aperture Illumination (Curve 4 b)

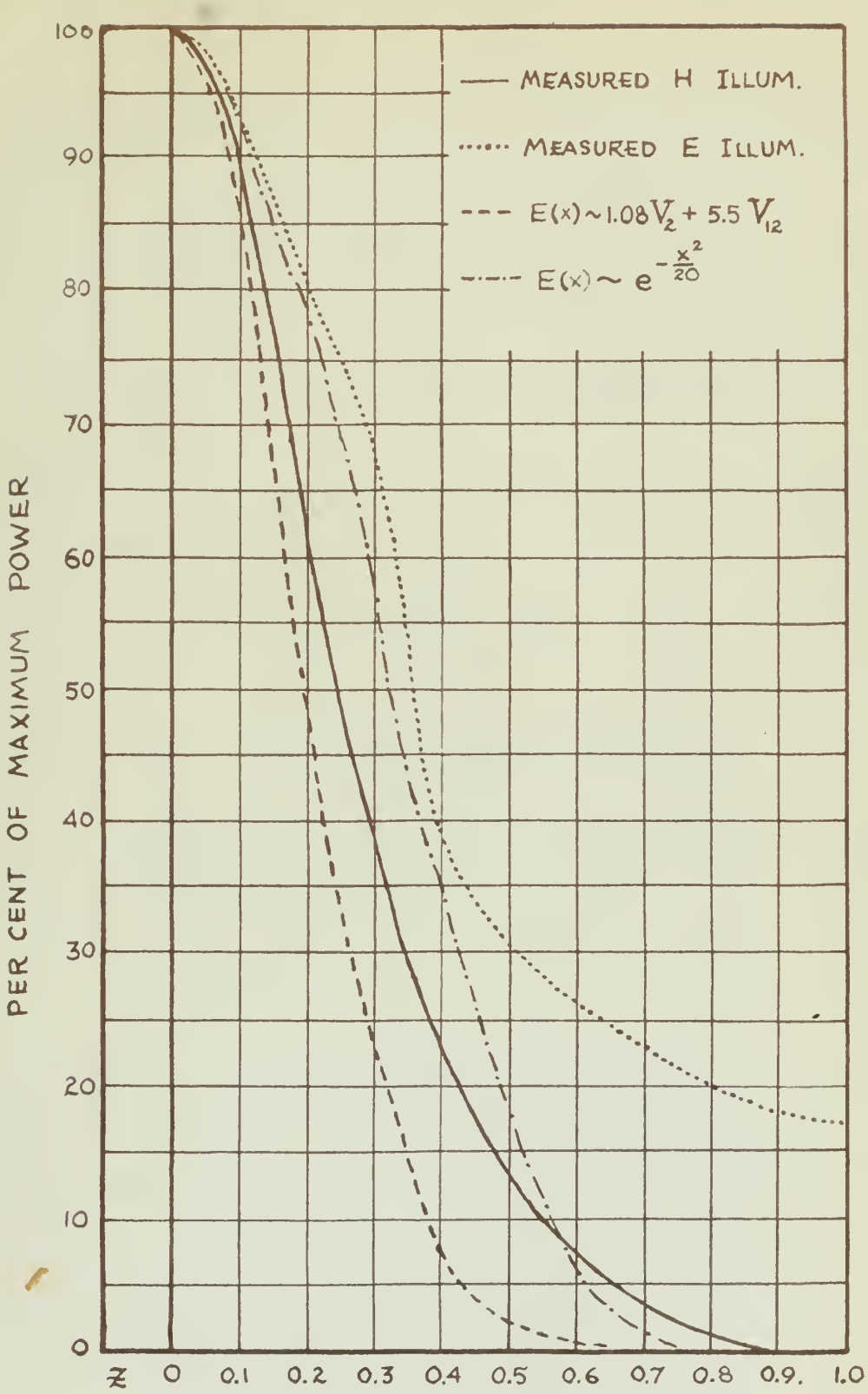
| z | Measured E** Power Distribution | $[I(z)]^2$ for $E(x) \sim 1.08V_2 - 5.5V_{12}$ | $[I(z)]^2$ for $E(x) \sim e^{-\frac{x^2}{20}}$ |
|---|---|---|---|
| 0.0 | 1.00 | 1.00 | 1.00 |
| 0.1 | 0.90 | 0.83 | 0.94 |
| 0.2 | 0.62 | 0.48 | 0.78 |
| 0.3 | 0.38 | 0.24 | 0.57 |
| 0.4 | 0.23 | 0.07 | 0.35 |
| 0.5 | 0.13 | 0.03 | 0.18 |
| 0.6 | 0.08 | 0.01 | 0.07 |
| 0.7 | 0.03 | | 0.02 |
| 0.8 | 0.02 | | |
| 0.9 | | | |
| 1.0 | | | |

* Measured Data is for parabolic reflector of 36 inch radius.

** Power distribution for the E plane as measured appears on Curve 4b.


ACTUAL MEASURED PATTERN
E(x) ~ 1.08 V2(x) + 5.5 V12(x)
E(x) ~ e^-x²/20
E(x), (NORMALIZED VALUE)
1.0
.8
.6
.4
.2
0
1
2
3
4
5
6
8
9
10
θ = 1°
θ = 2°
x
(x = ka sin θ ; ka = 191)
DISTANT FIELD PATTERNS

CURVE 4a


MEASURED H ILLUM.
MEASURED E ILLUM.
E(x)~1.08V2 + 5.5 V12
E(x)~ e^-x²/20
PER CENT OF MAXIMUM POWER
100
90
80
70
60
50
40
30
20
10
0
z
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9.
1.0
APERTURE ILLUMINATION

CURVE 4b

expanded to allow expression as a function of $(x^2)$:

$$E(x) \sim e^{-\frac{x^2}{20}} = 1 - \frac{x^2}{20} + \frac{x^4}{(20)^2 2!} + \cdots .$$

The expansion was then multiplied by $e^{-ixz}\,dx$ and integrated over the limits of 0 to $\infty$ by the use of formula (4.30).

$$\int_0^\infty e^{-ixz} E(x)\,dx \sim \int_0^\infty e^{-ixz}\left[1 - \frac{x^2}{20} + \frac{x^4}{(20)^2 2!} + \cdots\right] dx$$

$$= \frac{1}{[iz]} - \frac{2!}{20\,[iz]^3} + \frac{4!}{(20)^2 2!\,[iz]^5} + \cdots .$$

To secure a first approximation for demonstration purposes, equations (4.36) were set up with k = 0 and 1.

$$k = 0: \qquad 1 - \prod_n \frac{(n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots(\tfrac{1}{2})}{n!} = 0 ,$$

$$k = 1: \qquad -\frac{1}{10} - \prod_n \frac{(n-\tfrac{1}{2})(n-\tfrac{3}{2})\cdots(\tfrac{1}{2})(-\tfrac{1}{2})}{(n+1)!} = 0 .$$

These equations can be solved for n, obtaining $n = 4$, $\prod_4 = 5.63$, $I_4 = \frac{8}{G_a^2}$.

Thus the aperture illumination indicated is

$$I(z) = \frac{8}{G_a}(1 - z^2)^3 .$$

This aperture illumination, tabulated in (Table 2b) and plotted on (Curve 4b), also shows excellent conformity with the experimental results. The nature of the approximation must be noted, however. If the third term of the exponential expansion is considered, an "n" of 4 is not sufficient. In contrast to the use of the first method, two approximations enter into the final calculation. The expression of E(x) as an exponential is an approximation and the use of a limited number of terms, in this case only one, is also an approximation.

VI.5 The material of paragraph (VI.4) demonstrated the peculiar nature of the method of Chapter IV. And it more than ever convinces the author that the "mathematical device" of Chapter IV is part of a transform relationship. A possible relationship of this nature is suggested in paragraph (4.6). Early in the course of the work on this essay, the method of Chapter III was determined; and it was realized that a complete solution of the radiation problem was at hand. All that remained to be done was to express an arbitrary curve as a summation of V functions. A test method, similar to the Fourier-bessel work, was visualized, but a great deal of research failed to disclose even the possibility of such expression. It is now clear that such a hope was an absurdity. Since the V functions decay so rapidly with increasing values of x and no phase variations could exist because of the required symmetry about x = 0, it is logical to believe that only even functions of a general decaying nature could possibly be expressed. And it is now known that such functions can be expressed as a summation of V functions to as high a degree of accuracy as is desired. The fact that a completely arbitrary function cannot be so expressed, in no way detracts from the usefulness of the method, for functions expressing radiation patterns are not completely arbitrary. They will be of the type which the method can handle. Let us then make this statement. Any radiation pattern can be expressed as a summation of V functions, and hence any radiation problem can be completely solved. The degree of accuracy will depend on the number of terms one is willing to consider. The author believes an infinite number of terms would in all cases give complete accuracy.

VI.6 The statements made in the preceding paragraph may sound a bit ostentatious. However, their substantiation is contained in the body of the essay. Any possible aperture distribution is expressible as a function of $(1 - z^2)$, and any such function results in a distant field pattern expressible as a summation of V functions. The reverse must be true by reciprocity.

VI.7 Let us consider first the exponential curve of paragraph (VI.4). To a first approximation, a was found to be 4. The method of Chapter III clearly indicates this value belongs to $V_4(z)$. Then to a first approximation, the function $e^{-\frac{x^2}{20}}$ must be capable of expression by $V_4(z)$ alone. The tabulated comparisons of these two functions appear in Table 6b and their plots on Curve 6b. The two functions are definitely similar to the order which might be expected for a first approximation. An n of 4 seems exactly correct if only the first two terms of the exponential expansion are considered. The third term, however, clearly demonstrates the need for additional V functions in the expression. The ratio of the third term to the second term in the exponential expansion is 0.1, while the method of Chapter IV indicates a ratio of 0.20 for the single value of n=4. If these two conditions were to be satisfied, that is k = 0,1,2 in equation (4.36), two values of n would be obtained from the solution of the three simultaneous equations and the exponential curve should be more closely approximated. This is a difficult but possible procedure. Trial and error by comparing the ratios of terms of the binomial expansion of $(1 - z^2)^{n-\frac{1}{2}}$ and combinations of these ratios to the desired ratio in the expansion of the given function may be simpler than the solution of the simultaneous equations of (4.36). Actually, however, the same thing is being done in both cases, as explained in

paragraph (4.12). The first order V approximation would always suffice for an indication of solution.

VI.8 Another example considered was the exponential

$$E(x) = e^{-\frac{x^2}{44}} = 1 - \frac{x^2}{44} + \frac{x^4}{(44)^2 2!} + \cdots.$$

It was decided that only one term would be considered in the V expression since only a first approximation of the result was desired. Having made this decision the simultaneous equations (4.36) were set up with n = 0,1. Solving these equations, n was determined as 10. This meant that the single V function most like the exponential under consideration was $V_{10}(x)$. The two functions are plotted on (Curve 8c) and tabulated in (Table 8c). As can be seen, the correspondence between the two is again amazing.

VI.9 The only functions which have been considered thus far have been exponentials, but the method is by no means restricted to such functions. To demonstrate the versatility of the method, a parabola was selected as the pattern curve.

$$E(x) = 1 - \frac{x^2}{10}$$

The same method as used previously was used in this case. However, since no single n would satisfy equations (4.36) with k = 0 and 1, a combination was sought which would satisfy the relationship. The two values of n which in equal part did satisfy (4.36) were 1 and 3. This meant that

$$E(x) = 1 - \frac{x^2}{10} \sim V_1(x) - V_3(x).$$

The pattern function and its V function approximation, which again is a first approximation, are tabulated in (Table 8A) and plotted on

# TABLE 6

## APPROXIMATION OF FUNCTIONS BY V FUNCTIONS (Curve 5)

(A) $E(x) = 1 - \frac{x^2}{10}$ ; $E(x)_A = V_1(x) + V_3(x)$

| x | E(x) | $E(x)_A$ |
|---|---|---|
| 0.0 | 1.00 | 1.00 |
| 0.5 | 0.98 | 0.97 |
| 1.0 | 0.90 | 0.90 |
| 1.5 | 0.78 | 0.77 |
| 2.0 | 0.60 | 0.62 |
| 2.5 | 0.38 | 0.46 |
| 3.0 | 0.10 | 0.31 |

(B) $E(x) = e^{\frac{-x^2}{20}}$; $E(x)_A = V_4(x)$

| x | E(x) | $E(x)_A$ |
|---|---|---|
| 0 | 1.00 | 1.00 |
| 1 | 0.95 | 0.95 |
| 2 | 0.82 | 0.82 |
| 3 | 0.64 | 0.63 |
| 4 | 0.45 | 0.42 |
| 5 | 0.28 | 0.24 |
| 6 | 0.17 | 0.11 |
| 7 | 0.09 | 0.03 |

(C) $E(x) = e^{\frac{-x^2}{44}}$; $E(x)_A = V_{10}(x)$

| x | E(x) | $E(x)_A$ |
|---|---|---|
| 0 | 1.00 | 1.00 |
| 1 | 0.98 | 0.98 |
| 2 | 0.91 | 0.91 |
| 3 | 0.81 | 0.81 |
| 4 | 0.69 | 0.69 |
| 5 | 0.57 | 0.56 |
| 6 | 0.44 | 0.43 |
| 7 | 0.33 | 0.31 |
| 8 | 0.24 | 0.21 |
| 9 | 0.16 | 0.13 |
| 10 | 0.10 | 0.06 |

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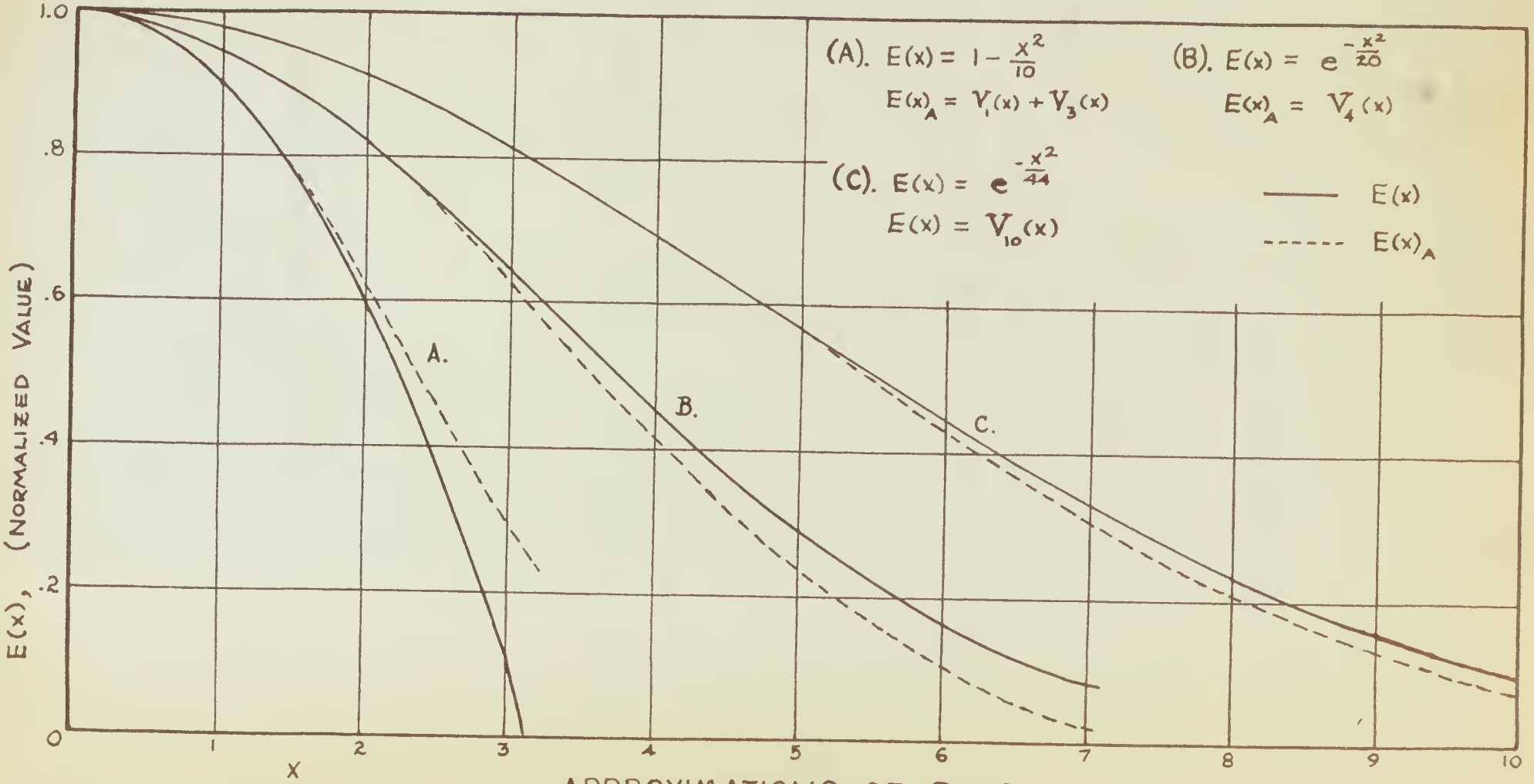


APPROXIMATIONS OF FUNCTIONS
CURVE 5

(Curve 5A). The correspondence again is good. And it is important to note from this example that the method has treated the parabola as if it converged to 0 as x increased. It is thus believed that any curve capable of expression as a function of $(x^2)$ can be approximated by this method as long as it is a decreasing function and its value remains greater than or equal to zero.

VI.10 The practical solution of the original problem is now complete. The method of Chapter IV tells which V functions express the distant field pattern and the method of Chapter III uses these selected V functions to determine the aperture illumination required for that given distant field pattern. Approximate results are easily obtainable in all cases, and these approximations are capable of refinement to perfection if the pattern curve is of continuous nature and if the mathematical complexity can be tolerated. In addition, the correspondence of the theoretical results and experimental data has been shown excellent. It is believed that the conformity of results to physical actuality and the ease of application will make the method here presented valuable whenever problems of this nature are considered.

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